Enhancing Engineering Students' Generic Thinking Skills through Puzzles, Paradoxes and Sophisms

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Previous Research:

Differential Equations with Multivalued Right-hand Side

(Differential Inclusions)

$$\frac{dx}{dt} \in F(t,x)$$

AVERAGING OF DIFFERENTIAL RELATIONS OF BELONGINGNESS WITH UNBOUNDED RIGHT-HAND SIDES

S. S. Klimchuk

UDC 517.928

In the present note we prove a relative estimate of closeness of solutions of the initial and the averaged differential relations of belongingness whose right-hand sides are not bounded.

Let us consider the differential relation

$$\dot{x} \in \varepsilon X(t, x), \qquad x(0) = x_0, \tag{1}$$

where x is an n-dimensional vector, ε is a small positive parameter, and X(t, x) is a multivalued 2π-periodic (with respect to t) mapping with values in $\mathcal{F}(R^n)$ – the space of nonempty closed subsets of R^n such that $0 \in X(t, x)\ \forall (t, x)$.

By a solution of the differential relation (1) we mean an absolutely continuous vector-valued function x(t), defined on the segment $\mathcal{J}$, such that $\dot{x}(t) \in \varepsilon X(t, x(t))$.

In correspondence with the differential relation (1) we put the averaged differential relation

$$\dot{\bar{x}} \in \varepsilon \overline{X}(\bar{x}), \quad \bar{x}(0) = x_0, \tag{2}$$

where $\overline{X}(x) = \frac{1}{2\pi}\int\limits_0^{2\pi} X(t, x)\,dt$.

The integral of an unbounded multivalued mapping is understood as the closure of the Aumann integral [3]

$$\int\limits_a^b F(t)\,dt = \overline{\left\{y \mid y = \int\limits_a^b f(t)\,dt,\ f(t) \in F(t)\right\}}.$$

Side by side with the differential relations (1) and (2), we consider the differential relations

$$\dot{x}_r \in \varepsilon X(t, x_r) \cap S_r(0), \qquad x_r(0) = x_0, \tag{3}$$

$$\dot{\bar{x}}_r \in \varepsilon \overline{X(t, \bar{x}_r) \cap S_r(0)}, \qquad \bar{x}_r(0) = x_0, \tag{4}$$

where $S_r(0)$ is the ball of radius r with center at 0 and r is a positive constant.

Let K(t) and $\overline{K}(t)$ denote the closures of the sections of families of solutions of the differential relations (1) and (2), respectively, and let $K_r(t)$ and $\overline{K}_r(t)$ denote the sections of the families of solutions of the differential relations (3) and (4), respectively. It will be shown below that $\lim\limits_{n\to\infty} K_n(t) = K(t)$ and $\lim\limits_{n\to\infty} \overline{K}_n(t) = \overline{K}(t)$.

At first, we prove the following lemma.

<u>LEMMA 1.</u> Let there be given a differential relation

$$\dot{x} \in F(t, x), \quad x(0) = x_0, \tag{5}$$

where the multivalued mapping $F(t, x) : [0, \infty) \times R^n \to \mathcal{F}(R^n)$ is continuous with respect to (t, x) [3], is convex, and satisfies the Lipschitz condition with constant λ with respect to x, i.e., $h(F(t, x'), F(t, x'')) \leqslant \lambda \|x' - x''\|$, where h(A, B) is the Hausdorff distance between the sets $A, B \subset \mathcal{F}(R^n)$, i.e., $h(A, B) = \max\{\sup\limits_{a\in A} \rho(a, B), \sup\limits_{b\in B} \rho(A, b)\}$, and $\rho(A, b) = \inf\limits_{a\in A}\|a - b\|$ is the distance of the point b from the set A. Moreover, let $0 \in F(t, x)\ \forall (t, x)$.

Odessa University. Translated from Ukrainskii Matematicheskii Zhurnal, Vol. 41, No. 3, pp. 389-391, March, 1989. Original article submitted September 18, 1986.

0041-5995/89/4103-0343$12.50

The families of the solutions of inclusions (2) and (6), and also their corresponding sections $\overline{R}(t)$ and $clR(t)$ accordingly, are compact sets [11].

Let us prove the first statement of the theorem and hence the validity of the inclusion

$$\overline{R}(t) \subset S_\eta(clR(t)). \tag{7}$$

Divide the interval $[0, L\varepsilon^{-1}]$ on the partial intervals with the points $t_i = \frac{Li}{m\varepsilon}$, $i = \overline{0, m}$, $m \in \mathbb{N}$. Let $\xi(t)$ be a solution of inclusion (2). Then there exists a measurable selector $v(t) \in \overline{X}(\xi(t))$ such that

$$\xi(t) = \xi(t_i) + \varepsilon \int_{t_i}^{t} v(\tau)\mathrm{d}\tau, \quad t \in [t_i, t_{i+1}], \qquad \xi(0) = x_0. \tag{8}$$

Consider the function

$$\xi^1(t) = \xi^1(t_i) + \varepsilon v_i(t - t_i), \quad t \in [t_i, t_{i+1}], \qquad \xi^1(0) = x_0, \tag{9}$$

where vector v_i satisfies the condition

$$\left\| \frac{L}{m\varepsilon} v_i - \int_{t_i}^{t_{i+1}} v(t)\mathrm{d}t \right\| = \min_{v \in \overline{X}(\xi^1(t_i))} \left\| \frac{L}{m\varepsilon} v - \int_{t_i}^{t_{i+1}} v(t)\mathrm{d}t \right\|. \tag{10}$$

The vector v_i exists and is unique in term of the compactness and convexity of the set $\overline{X}(\xi^1(t_i))$ and the strong convexity of the function being minimized.

Let $\delta_i = \|\xi(t_i) - \xi^1(t_i)\|$. As

$$\|\xi(t) - \xi(t_i)\| = \varepsilon \left\| \int_{t_i}^{t} v(\tau)\mathrm{d}\tau \right\| \le \varepsilon M(t - t_i) \le \frac{ML}{m}, \tag{11}$$

then

$$\begin{aligned} \|\xi(t) - \xi^1(t_i)\| &\le \|\xi(t) - \xi(t_i)\| + \|\xi(t_i) - \xi^1(t_i)\| \\ &\le \delta_i + \varepsilon M(t - t_i), \end{aligned}$$

$$h(\overline{X}(\xi(t)), \overline{X}(\xi^1(t_i))) \le \lambda[\delta_i + \varepsilon M(t - t_i)], \quad t \in [t_i, t_{i+1}]. \tag{12}$$

$$\begin{aligned} &\le \left(1 + \frac{\lambda L}{m}\right)^{i+1} \delta_0 + \frac{\lambda M L^2}{2m^2} \sum_{k=0}^{i} \left(1 + \frac{\lambda L}{m}\right)^k \\ &\le \frac{ML}{2m}\left[\left(1 + \frac{\lambda L}{m}\right)^{i+1} - 1\right] \le \frac{ML}{2m}(\mathrm{e}^{\lambda L} - 1), \\ &i = \overline{0, m-1}. \end{aligned} \tag{14}$$

As

$$\|\xi^1(t) - \xi^1(t_i)\| = \varepsilon \|v_i\|(t - t_i) \le \frac{ML}{m}, \tag{15}$$

then from (11) and (14) it follows that

$$\begin{aligned} \|\xi(t) - \xi^1(t)\| &\le \|\xi(t) - \xi(t_i)\| + \|\xi(t_i) - \xi^1(t_i)\| \\ &\quad + \|\xi^1(t_i) - \xi^1(t)\| \le \frac{ML}{2m}(\mathrm{e}^{\lambda L} + 3). \end{aligned} \tag{16}$$

From condition (2) of the theorem it follows that for any $\eta_1 > 0$ and fixed m the following inequality holds

$$h\left(\frac{\varepsilon m}{L} \int_{t_i}^{t_{i+1}} X(t, \xi^1(t_i))\mathrm{d}t, \overline{X}(\xi^1(t_i))\right) \le \eta_1. \tag{17}$$

Hence, there exists such measurable selector $v^i(t) \in X(t, \xi^1(t_i))$, $i = \overline{0, m-1}$ that

$$\left\| \frac{\varepsilon m}{L} \int_{t_i}^{t_{i+1}} [v^i(t) - v_i]\mathrm{d}t \right\| \le \eta_1. \tag{18}$$

Consider the family of functions

$$\begin{aligned} &x^1(t) = x^1(t_i) + \varepsilon \int_{t_i}^{t} v^i(\tau)\, \mathrm{d}\tau, \quad t \in [t_i, t_{i+1}], \\ &x^1(0) = x_0. \end{aligned} \tag{19}$$

From (18), (19) and (9) it follows that

$$\|x^1(t_i) - \xi^1(t_i)\| \le \|x^1(t_{i-1}) - \xi^1(t_{i-1})\|$$

1.1. The full averaging scheme

1.1.1. The averaging on the finite interval

Consider the differential inclusion

$$\dot{x} \in \varepsilon X(t,x), \qquad x(0) = x_0, \tag{1}$$

where $t \in \mathbb{R}^+$ is time, $x \in \mathbb{R}^n$ is a phase vector, $\varepsilon > 0$ is a small parameter, $X : \mathbb{R}^+ \times \mathbb{R}^n \to comp(\mathbb{R}^n)$ is a multivalued mapping, $comp(\mathbb{R}^n)(\mathrm{conv}(\mathbb{R}^n))$ is the set of all nonempty compact (and convex) subsets of $\mathbb{R}^n$ with Hausdorff metric:

$$h(A,B) = \min\{r \geq 0 : A \subset B + S_r(0), B \subset A + S_r(0)\}.$$

$S_r(a)$ is the ball in $\mathbb{R}^n$ with radius $r \geq 0$ and center in the point $a \in \mathbb{R}^n$.

Let us associate with inclusion (1) the following averaged differential inclusion

$$\dot{\xi} \in \varepsilon \overline{X}(\xi), \qquad \xi(0) = x_0, \tag{2}$$

where

$$\overline{X}(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T X(t,x)\,dt. \tag{3}$$

Here the integral of the multivalued mapping is understood in Aumann sense [9] and the convergence in sense of the Hausdorff metric.

Theorem 1 *([3,7]). Let in the domain $Q = \{t \geq 0, x \in D \subset \mathbb{R}^n\}$ the following hold:*

(1) *the mapping $X(t,x)$ is continuous, uniformly bounded with constant M, satisfies the Lipschitz condition in x with constant λ;*
(2) *limit (3) exists uniformly with respect to x in the domain D;*
(3) *for any $x_0 \in D' \subset D$ and $t \geq 0$ the solutions of inclusion (2) together with a ρ-neighborhood belong to the domain D.*

Then for any $\eta \in (0,\rho]$ and $L > 0$ there exists $\varepsilon^0(\eta, L) > 0$ such that for all $\varepsilon \in (0,\varepsilon^0]$ and $t \in [0, L\varepsilon^{-1}]$ the following statements hold:

(1) *for any solution $\xi(t)$ of inclusion (2) there exists a solution $x(t)$ of inclusion (1) such that*

$$\|x(t) - \xi(t)\| \leq \eta; \tag{4}$$

(2) *for any solution $x(t)$ of inclusion (1) there exists a solution $\xi(t)$ of inclusion (2) such that inequality (4) holds.*

Thereby,

$$h(\overline{R}(t), \mathrm{cl}R(t)) \leq \eta, \tag{5}$$

where $\overline{R}(t)$ is the section of the family of the solutions of the averaged inclusion, $\mathrm{cl}R(t)$ is the closure of the section of the family of the solutions of the initial inclusion.

Current Research: Mathematics Education

- Using counterexamples, paradoxes and sophisms to enhance students' understanding of undergraduate mathematics

- Effective teaching of mathematical modelling and applications to undergraduate students

- Analysing the transition from secondary to university education in mathematics

- Influence of attention on mathematical knowledge and assessment http://www.ams.org/msc/

2000 Mathematics Subject Classification (MSC)

97: Mathematics Education is an area newly added to the MSC effective with the 2000 revision, but of a long heritage. Topics of discourse cover all levels of mathematics education from pre-school to university level, and can focus on the student (through educational psychology, for example), the teacher (continuing development or assessment), the classroom (and the books and technology used in the classroom), or the larger system (policy analysis, cross-cultural comparisons, and so on). Analysis can range from small case studies to large statistical surveys; perspectives range from the fairly philosophical to the clinical.

SEOUL ICM 2014

SEOUL ICM 2014

International Congress of Mathematicians

August 13 - 21, 2014
Coex , Seoul , Korea

Scientific Program (Sections)

1. Logic and Foundations
2. Algebra
3. Number Theory
4. Algebraic and Complex Geometry
5. Geometry
6. Topology
7. Lie Theory and Generalizations
8. Analysis and its Applications
9. Dynamical Systems and Ordinary Differential Equations
10. Partial Differential Equations
11. Mathematical Physics
12. Probability and Statistics
13. Combinatorics
14. Mathematical Aspects of Computer Science
15. Numerical Analysis and Scientific Computing
16. Control Theory and Optimization
17. Mathematics in Science and Technology
18. Mathematics Education and Popularization of Mathematics
19. History of Mathematics

The 12th Engineering Mathematics and Applications Conference (EMAC-2015), Adelaide, Australia

Aust MS
Australian Mathematical Society Inc.

ANZIAM
Australian & New Zealand Industrial & Applied Mathematics

Topics for EMAC2015 include, but are not limited to:

- Biomedical
- Chemical engineering
- Computational fluid dynamics
- Differential equations
- Dynamical systems
- Engineering mathematics
- **Engineering mathematics education**
- Environment
- Financial engineering
- Mathematical biology
- Non-linear systems
- Operations research
- Optimisation
- Stochastic and statistical modelling

53^{rd} Annual Meeting of the Australian Mathematical Society

Special sessions

- Algebra and Number Theory
- Analysis and PDEs
- Applied Differential Equations
- Combinatorics
- Computational Mathematics
- Fundamental Statistics
- General Session
- Geometric Analysis
- Geometry and Topology
- History and Philosophy of Mathematics
- Interpolation in Mechanics
- Logic
- **Mathematics Education**
- Mathematical Physics
- Mathematical Psychology
- New Applications of Mathematics
- Noncommutative Geometry and Operator Algebras
- Optimal Control
- Optimization: Theory and Methods
- Probability: Theory and Applications
- Random Processes
- Statistical Applications
- Topological and Symbolic Dynamical Systems
- Topological Groups

PBRF Guidelines 2012

Mathematical and Information Sciences and Technology

Applied mathematics includes the development of, the analysis of, and the solution or approximate solution of mathematical models including those arising in physical, geophysical, marine and life and health sciences, engineering and technology; it also includes the development and application of mathematical theories and techniques that further these objectives. The Mathematical and Information Sciences and Technology panel will also consider operations research and optimisation, including deterministic and stochastic models and solution methods. This subject area also includes mathematics education.

THE UNIVERSITY OF AUCKLAND
NEW ZEALAND
Te Whare Wānanga o Tāmaki Makaurau

Department of Mathematics

https://www.math.auckland.ac.nz/en.html

Research Groups

The Department of Mathematics has four major research groups; Applied Mathematics, Algebra and Combinatorics, Analysis and Topology, and Mathematics Education.

ARIZONA STATE UNIVERSITY

School of Mathematical and Statistical Sciences

https://math.asu.edu/

Four Areas of Math and Statistics, Under One Roof

Whether you want to apply math to science or engineering, search for a deeper understanding of theoretical mathematics, develop effective ways to teach mathematics, or make sense of data with statistics, we have something for you.

Applied Mathematics

Theoretical Mathematics

Mathematics Education

Statistics

STEM Education is Part of STEM

STEM is a group of inter-related subjects that are crucial for the knowledge economy and prosperity of the country

The guidelines of the National Science Foundation, USA:

- Chemistry
- Computer and Information Technology Science
- Engineering and Geosciences
- Life Sciences
- Mathematical Sciences
- Physics and Astronomy
- STEM Education and Learning Research

STEM Education is Part of STEM

STEM eligible degrees in the USA Immigration:

Physics, Actuarial Science, Chemistry, Mathematics, Applied Mathematics, Statistics, Computer Science, Computational Science, Psychology, Biochemistry, Robotics, Computer Engineering, Electrical Engineering, Electronics, Mechanical Engineering, Industrial Engineering, Information Science, Civil Engineering, Aerospace Engineering, Chemical Engineering, Astrophysics, Astronomy, Optics, Nanotechnology, Nuclear Physics, Mathematical Biology, Operations Research, Neurobiology, Biomechanics, Bioinformatics, Acoustical Engineering, Geographic Information Systems, Atmospheric Sciences, Educational/Instructional Technology, Software Engineering, and Educational Research

STEM Education

Better education in STEM subjects leads to stronger economy, defence and security of the country

"Countries that out-educate us now will out-perform us in the future"   President Obama

Why?

- Paradoxes
- Provocations
- Sophisms
- Counterexamples
- Puzzles
- Mistakes in textbooks
- Misuses of maths

Maths is “too dry and boring”

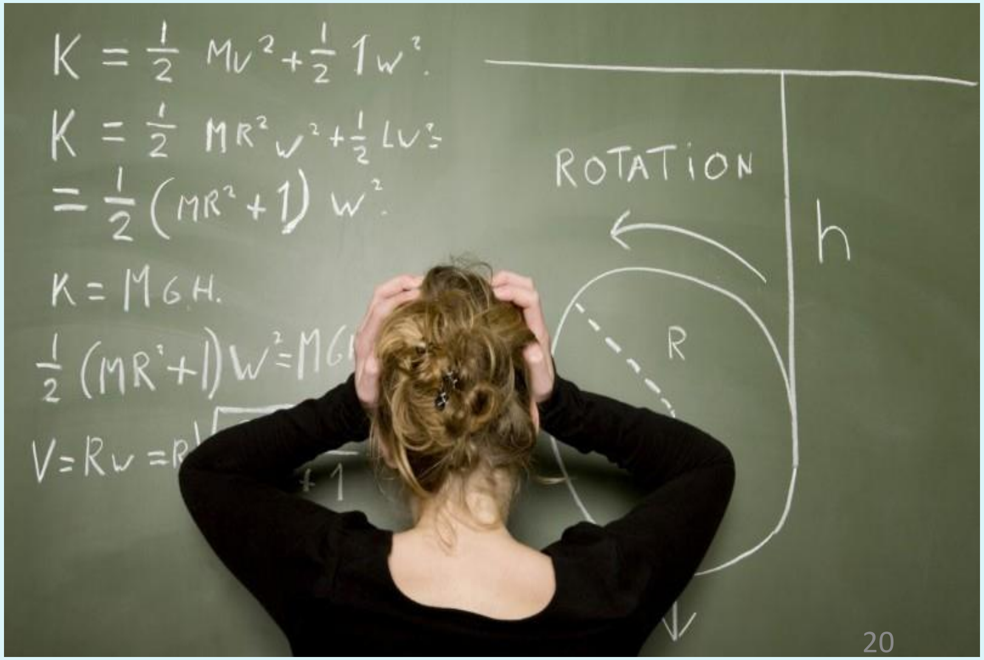
Emotional disengagement and academic disinterest

If the students are interested the rest is easy

“A student is not a vessel to be filled, but a fire to be lit”

Plutarch

Perceptions of Maths



Maths is a Way of Thinking

- Looking for evidence and proof
- Not taking anything for granted
- Logical reasoning/problem solving
- Conceptual understanding
- Critical/analytical thinking
- Paying attention to details

Useful skills in other areas of life!

Finding a Pattern is not Enough!

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \mapsto$$

???

$$\lim_{x \to 8} \frac{1}{(x-8)^2} = \infty$$

!!!

Why is Attention to Details Important?

- All conditions of a theorem;
- The locality (interval/point) of a statement;
- The properties of the function involved;
- The shape of the brackets;
- The word order;
- Etc.

Why bother?

Example: Singularities

Engineers must learn about singularities - special points where a given mathematical object is not defined or not "well-behaved" in some particular way (e.g. discontinuity, non-differentiability)

In real life, singularities are catastrophes

Puzzles vs routine problems

- A puzzle is a non-routine, non-standard problem presented in an entertaining way
- *Simplicity:* Easy to state and remember and looks deceptively simple
- *Surprise:* Teases by a surprising solution and an unexpected counterintuitive answer
- *Entertainment:* Fun to solve

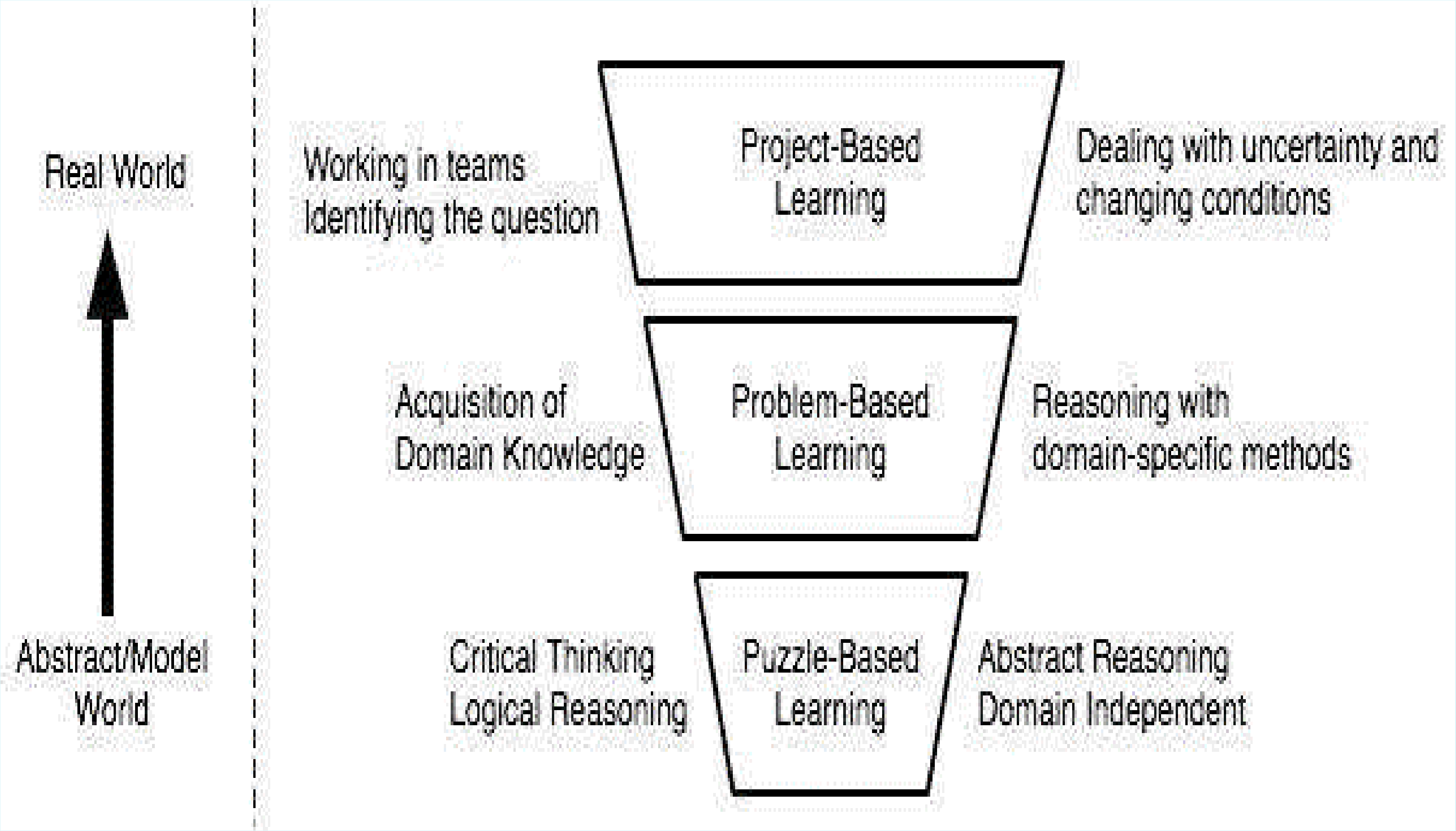
Edutainment = Education + Entertainment

- **Paradox** - looks invalid but is in fact true
- **Provocation** - looks routine but has a catch
- **Sophism** - looks correct but contains a deliberate flaw

Puzzle-Based Learning (PzBL)

- Developed by Z. Michalewicz (2008, 2014)
- Michalewicz, Z. & Michalewicz, M. (2008). *Puzzle-Based Learning: An introduction to critical thinking, mathematics, and problem solving*. Hybrid Publishers.
- Meyer, E.F., Falkner, N., Sooriamurthi, R., Michalewicz, Z. (2014). *Guide to Teaching Puzzle-Based Learning*. Springer.
- Criteria for a puzzle: independence (domain free); generality; simplicity; eureka factor; entertainment factor

Theoretical Considerations



Relationship between different PBL concepts (Falkner et al., 2012a)

Example: A chocolate bar

A rectangular chocolate bar consists of $m \times n$ small rectangles and you wish to break it into its constituent parts. At each step, you can only pick up one piece and break it along any of its vertical or horizontal lines. How should you break the chocolate bar using the *minimum* number of steps (breaks)? What is the *minimum* number?

Puzzle-Based Learning courses

- PzBL concept/approach has a long history
- First maths puzzles in Sumerian texts - 2500 BC
- Alcuin, an English scholar born around 732 AD, "Problems to Sharpen the Young" 50 puzzles

- New: Developing PzBL ***academic*** courses for 1st year maths, computer science and engineering students – ***compulsory*** at some universities

Benefits for the Students - 1

1. **Generic:**

- Engage students' emotions, creativity and curiosity
- Enhance problem-solving, critical thinking and generic thinking skills
- Improve lateral thinking "outside the box"
- Increase motivation and the retention rate

Benefits for the Students - 2

2. Professional: illustrate many powerful problem solving strategies

"Many engineering problems are puzzlelike. Pieces of the puzzle are provided to engineers in the form of user/customer requirements, technological constraints, professional or industrial codes, and market realities. The engineer must then craft a product or process that either meets all these (often conflicting) demands or else provides partial solutions, with clear justification of the tradeoffs made when meeting all of the specifications is not possible…Puzzling problems are, of course, plentiful in the research arena, regardless of the discipline."

Parhami (2008)

Benefits for the Students - 3

3. Employability: practice for job interviews

Many companies use puzzles at their job interviews to evaluate candidate’s problem solving skills and select best of the best.

“Now more than ever, an education that emphasizes general problem solving skills will be important”.

Bill Gates

Puzzles at Job Interviews

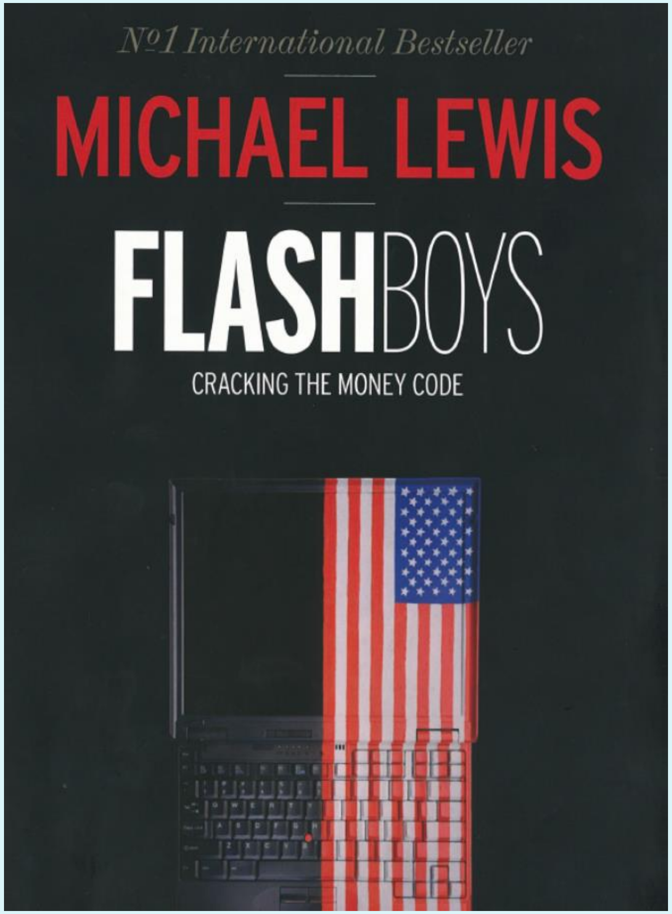
"The goal of Microsoft's interviews is to assess a general problem-solving ability rather than a specific competence. At Microsoft, and now at many other companies, it is believed that there are parallels between the reasoning used to solve puzzles and the thought processes involved in solving real problems of innovation. You have to hire for general problem-solving capacity."

Poundstone, W. (2004). How would you move Mount Fuji?

Real Examples

From the book "Flash Boys" by Michael Lewis, 2014

Hiring a programmer
for a financial company
on Wall Street, New York
with the annual salary
of 270,000 USD



Real Examples: Question 1

Question 1: Is 3599 a prime number?

This is where the boring formula from algebra

$$a^2 - b^2 = (a - b) \times (a + b)$$

can help you to become rich ☺

$$3599 = (3600 - 1) = (60^2 - 1^2) = (60 - 1)\,(60 + 1) = 59 \times 61$$

Real Examples: Question 2

sions. “He says there is a spider on the floor, and he gives me its coordinates. There is also a fly on the ceiling, and he gives me its coordinates as well. Then he asked the question: Calculate the shortest distance the spider can take to reach the fly.” The spider can’t fly or swing; it can only walk on surfaces. The

Real Examples: Question 2

shortest path between two points was a straight line, and so, Serge figured, it was a matter of unfolding the box, turning a three-dimensional object into a two-dimensional surface, then using the Pythagorean theorem to calculate the distances. This took him several minutes to work out; when he was done, Davidovich offered him a job at Goldman Sachs. His starting salary plus bonus came to $270,000.

Benefits for the Students - 4

4. Fun Short Breaks: (from a heavy content)

Recent example from my class this year:

Question 3. Can you see any other benefits for you in solving the puzzles?

a) Yes What are they?

b) No Why?

Response: “Maybe – too tired to think”

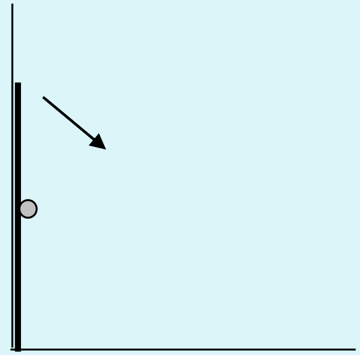
Two Pilot Studies – 2015, 2016

- Second year engineering mathematics course
- 6 weeks: 2 puzzles a week during a break between 2 consecutive lectures
- Voluntary participation
- 2015: 62 responses out of 65 (response rate 95%)
- 2016: 93 responses out of 97 (response rate 96%)

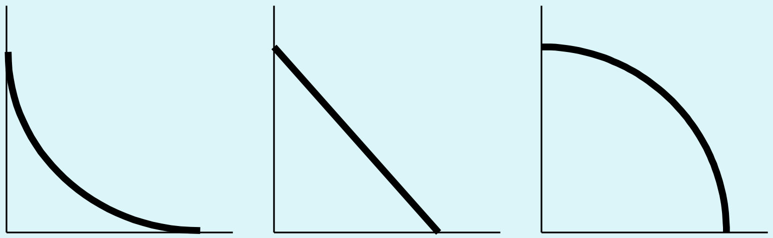
Examples of the Puzzles Given to the Students

Puzzle 1: A Cat on a Ladder

Imagine a cat sitting half way up a ladder that is placed almost flush with a wall.

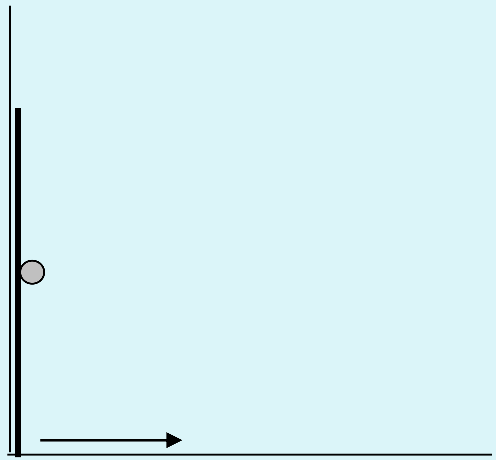


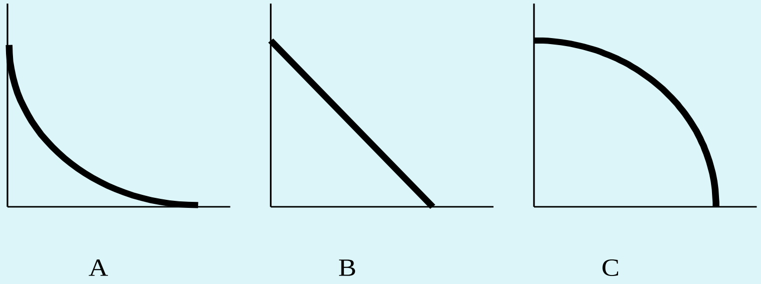
a) If the ladder falls what will the trajectory of the cat be? A, B or C?



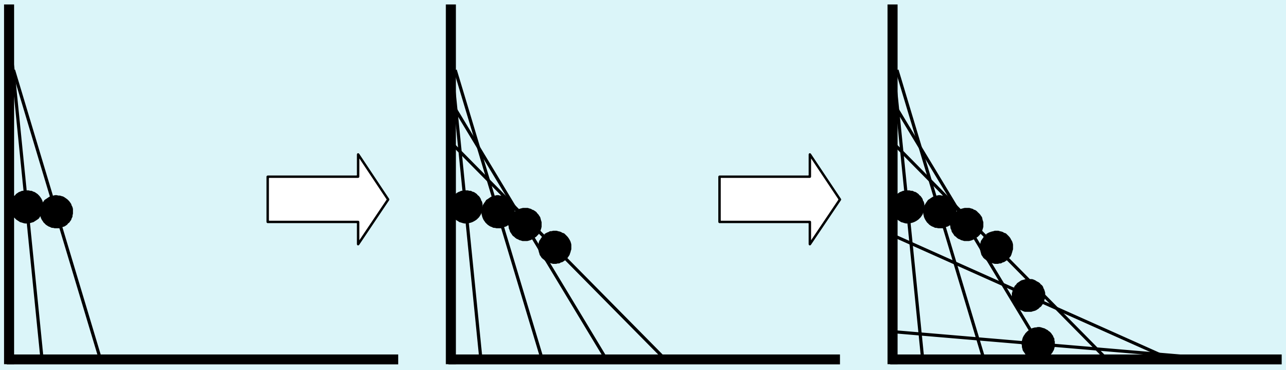
A B C

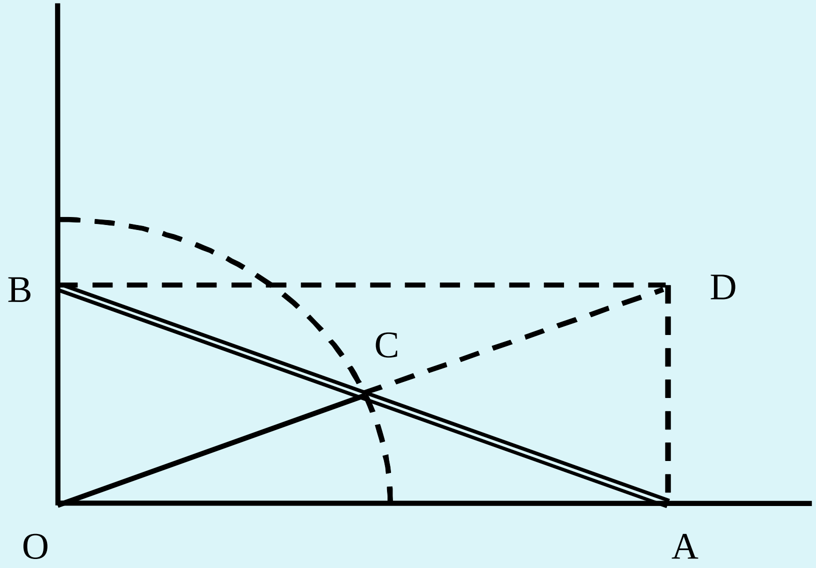
b) Now the *base* is pulled away with the top of the ladder retaining contact with the wall. What will the trajectory of the cat be? A, B or C?





Still C) !!!

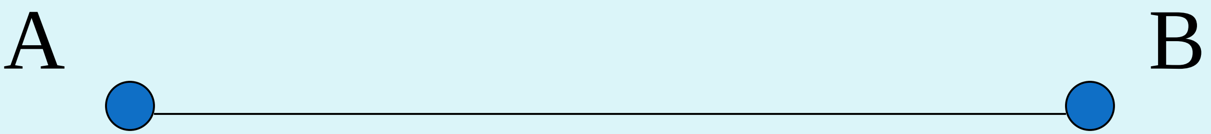




Puzzle 2: Average Speed

You drive a car at a constant speed of 40 km/h from **A** to **B**, and on arrival at **B** you return immediately to **A**, but at a higher speed of 60 km/h.

What was your average speed for the whole trip?



Puzzle 2: Average Speed

Average speed = Total distance / Total time

Try the distance between A and B as 120 km.

48 km/h

Puzzle 2: Average Speed

Suppose that you go from **A** to **B** at a constant speed of 40 km/h. What should your constant speed be for the return trip from **B** to **A** if you want to obtain the average speed of 80 km/h for the whole trip?

Puzzle 2: Average Speed

It is impossible! Even if you drive back from

B to A at the speed of light your average speed for the

whole trip still would be less than 80 km/h.

Consider the distance between A and B as 40 km.

Average speed = Total distance / Total time

Average speed of 80 km/h = 80 km / (1 h + … h)

Paradox: Torricelli's Trumpet

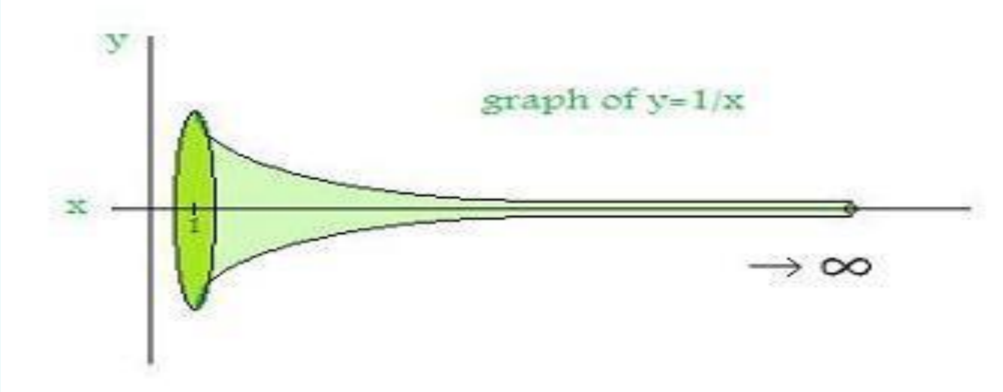
There is not enough paint in the world to paint the area bounded by the curve $y = \frac{1}{x}$, the x-axis, and the line $x = 1$:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

However, one can rotate the area around the x-axis and the resulting solid of revolution would have a finite volume:

$$\pi\int_1^\infty \frac{1}{x^2}\,dx = -\pi \lim_{b\to\infty}\left(\frac{1}{b} - \frac{1}{1}\right) = \pi.$$

One can fill the solid with π cubic units of paint and thus cover the cross-section area with paint.



Paradox: Torricelli's Trumpet

We should differentiate the "mathematical" universe from the "physical" universe. From a mathematical point of view one "abstract" drop of paint is enough to cover any area, no matter how large. One just needs to make the thickness of the cover very thin. Let's say we have 1 drop of paint of the volume of 1cm^3 and we need to cover a square plate of the size x by x cm. Then the (uniform) thickness of the cover will be $\frac{1}{x^2}$ cm. If x = 100cm then the thickness is $\frac{1}{10000}$ cm. If $x \to \infty$ then the area $x^2 \to \infty$ and the thickness $\frac{1}{x^2} \to 0$. But at any stage the volume is $x^2 \times \frac{1}{x^2} = 1$ cm^3. So mathematically you can cover any infinite area with any finite amount of paint, even with a single drop. In reality such infinite areas don't exist, nor can one make the cover infinitely thin.

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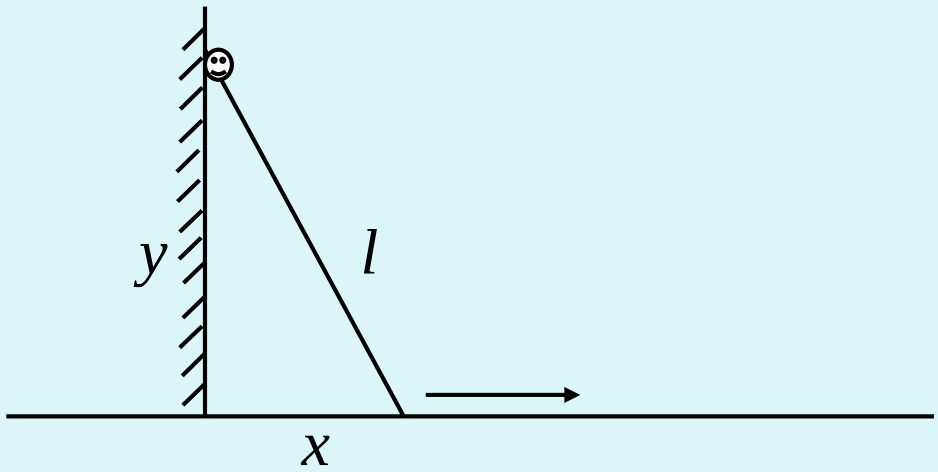
Sophism: An Infinitely Fast Fall

Imagine a cat sitting on the top of a ladder leaning against a wall. The bottom of the ladder is pulled away from the wall horizontally at a uniform rate. The cat speeds up, until it's falling **infinitely** fast.



‘Proof’

$y(t)=\sqrt{l^2-x^2(t)}$ where $x(t)$ and $y(t)$ are the horizontal and vertical distances from the ends of the ladder to the corner at time t. Since the ladder is pulled uniformly x' is a constant.

$$y' = -\frac{xx'}{\sqrt{l^2-x^2}} \qquad \lim_{x\to l} y' = \lim_{x\to l}\left(-\frac{xx'}{\sqrt{l^2-x^2}}\right) = -\infty$$

A Wrong Assumption

The ‘proof’ assumes that the ladder maintains contact with the wall while being pulled. This model is simply not true. If all forces involved are considered it can be shown that at one stage the top of the ladder will loose contact and be pulled away from the wall. From that moment the relationship $y = \sqrt{l^2 - x^2}$ is no longer true, since we don’t have a right-angled triangle.

Pilot Study 1 - Questionnaire

Question 1. Do you feel confident solving puzzles?

a) Yes b) No Please give the reasons:

Question 2. Can solving puzzles enhance your problem solving skills?

a) Yes In which way?

b) No Why not?

Question 3. From you point of view, what are the main differences between puzzles and routine problems/questions?

Pilot Study 1 - Results

Question 1. Do you feel confident solving puzzles? Please give the reasons.

Yes – 69%

- I have fair idea at times
- Because I am smart
- Because I can
- I am good at problem solving
- I love solving puzzles

No – 31%

- I overthink the problems
- I need more examples to understand the way how to do the question
- Too hard; can get confusing
- I feel that there will always be a catch
- I tend to overcomplicate everything
- I am constrained by knowledge taught by school system

Pilot Study 1 - Results

Question 2. Can solving puzzles enhance your problem solving skills?

Yes – 98% In which way?

- Helps your brain to think more logically and becomes challenging
- Make you look at problems from different angles
- Broadens mind for alternative solutions
- Think in a different perspective, outside the box
- Showing that thinking differently can have amazing results
- Make me think creatively, not always relying on conventional/trained ways of problem solving
- Puzzles place an emphasis on HOW you tackle the problem
- Ability to think with multiple perspectives
- It allows me to come to a solution faster
- Need more puzzles in all maths papers ☺

No – 2% Why not?

- Too different

Pilot Study 1 - Results

Question 3. From you point of view, what are the main differences between puzzles and routine problems/questions?

- Puzzles are more fun to solve; more enjoyable and interesting
- Puzzles are more challenging because of the flexibility in approaching
- Puzzles require creative thinking and more careful reading
- Puzzles add a bit more variety; are more tricky, freshen up your mind
- Puzzles require more insight, creativity; more thinking and novel solutions
- A puzzle requires us to throw away those old/stubborn stuff in my brain in order to solve it
- Puzzles relate to more realistic things
- Puzzles are exciting and help to keep me alert
- Puzzles set a more fun environment compared to routine problems
- Puzzles test your problem solving skills and routine problems are testing if you can follow problems

Pilot Study 2 - Questionnaire

Question 1. Can solving puzzles enhance your problem solving skills?

a) Yes — In which way?

b) No — Why not?

Question 2. Can solving puzzles enhance your generic thinking skills?

a) Yes — In which way?

b) No — Why not?

Question 3. Can you see any other benefits for you in solving puzzles?

a) Yes — What are they?

b) No — Why?

Pilot Study 2 - Results

Question 1. Can solving puzzles enhance your problem solving skills?

Yes – 97%   In which way?

- Gets you to think outside the box (>30)
- Being able to approach questions differently (>20)
- When it comes to solving practical problems in life (>20)
- Promotes learning using realistic situations (>10)
- You develop a more logically wired brain and you think about problems more open-mindedly (>10)
- Forces you to think creatively (it develops creative thinking which is important when facing non standard exam questions) (>5)
- These problems relate to problems engineers may come across in real life, and solving them is good experience (>5)

No – 3%

Pilot Study 2 - Results

Question 2. Can solving the puzzles enhance your generic thinking skills?

Yes – 97%

- By making you think about different situations in alternative ways (>20)
- It is about learning to think logically and methodically (>10)
- It will make students think about their theoretical solution and compare it with real world situations (>10)
- Use creative part of brain to decide on best answer (they are just really good at getting you to think creatively) (>5)
- You tend to see everyday life as puzzles you can solve (>5)
- Makes you think in practical ways
- Require us to step back and think in a broader scale

No – 3%

- They don't help learning
- Puzzles can be confusing

Pilot Study 2 - Results

Question 3. Can you see any other benefits for you in solving the puzzles?

Yes – 82%   What are they?

- It is a kind of fun break from the lecture which can help me concentrate (a break from the serious stuff; creating a fun learning environment; makes maths fun; allows a mini pause; it is a nice break from the current material, and acts as a nice mental break during lectures; good way to escape doing triple integrals; helps relax mind to perform better) (>20)
- Gets you thinking and involved (engage the class more than standard maths questions) (>10)
- You can use these new and different methods of thoughts in other subjects and aspects of life (>5)
- Relates to real life (life is all about solving puzzles) (>5)
- Helps with visualization and general articulation of problems (3)
- Possibly social interaction/team working (3)

Pilot Study 2 - Results

Question 3. Can you see any other benefits for you in solving the puzzles?

Yes – 82% What are they? (continued)

- It gives you a sense of accomplishment (learning patience/perseverance)
- You learn to filter out the useful, relevant info from the pointless
- Not being constrained in plugging numbers into equations
- It will come later in practical situations

No – 18% Why?

- The main benefits are given in questions 1 and 2
- Already covered in other subjects
- Don't learn basic maths techniques
- Puzzles may tend to throw off people even though it's an excellent method of testing
- Maybe – too tired to think

The Most Unexpected Comment

Question 3. Can you see any other benefits for you in solving the puzzles?

a) Yes — What are they?

Solving puzzles are great. But a puzzle called 'girl' can never be solved. There is no mathematical or theoretical solution to this particular puzzle called 'girl'.

b) No — Why?

Conclusions from both Studies

- Both Studies: Almost all participants (98% and 97%) believe that solving puzzles enhances their problem solving skills
- Study 1: The students' comments on the differences between a puzzle and a routine question were consistent with the views of authors of books and articles on Puzzle-Based Learning
- Study 2: Almost all participants (97%) believe that solving puzzles enhances their generic thinking skills
- Study 2: The vast majority (82%) indicated other benefits for them apart from enhancing generic and problem solving skills. The most common was "a nice fun break from the serious stuff" that "helps relax mind to perform better".
- My Concern in Both Surveys: Students would complain about taking time for puzzles from the main material

Nobody complained (!)

Future (Rigorous) Research

- Evaluate the relationship between the ability in solving puzzles and course performance
- Investigate the effect of using puzzles on student engagement and attendance
- Analyse the impact of using puzzles on a student decision to continue their study (retention)
- Assess students’ attitudes using attitudes scales
- Measure the cognitive dimension of the student engagement (investment in learning, perseverance in the face of challenges, and use of deep rather than superficial strategies)

Thank you for your attention

and

what about the Chocolate Bar Puzzle?

$$\cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} \ldots = 0$$

$$1 - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} \ldots = 1$$

So, 0 = 1 !!!

How about that:

$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \ldots = S$

$1 - (1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots) = S$

$1 - S = S$

$2S = 1$

$S = ½$

Euler and Leibnitz did believe so!!!

Final Thanks!