Survey of the Field of Empirical Research on Scientific Methods in STEM

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Scientific Inquiry in Mathematics - Theory and Practice

A STEM Perspective

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Preface

Albert Einstein wondered, "How can it be that mathematics, being after all a product of human thought which is independent of experience, is so admirably appropriate to the objects of reality?"[1]

January 27, 1921, address to the Prussian Academy of Science, Berlin.

Objective of the Book

A gap between problem-solving in mathematics and scientific inquiry in science is beyond controversy. Attempts to elevate the gap are made and this book aspires to be one of such attempts. The book is to promote transdisciplinary STEM learning experiences that support the thesis that exploring mathematics concepts using scientific methods can help to merge the two methodologies. Furthermore, it is hoped that such learning settings are to develop students' mathematical reasoning skills and serve as a means to improve students' STEM readiness.

The objective of this book is to propose a theoretical framework and multidisciplinary modeling activities of what STEM learning in the twenty-first century classroom might look like.

This book draws on a diverse literature from international STEM education community as well as from engineering, science, and mathematics education communities. It synthesizes the research findings that lead to formulating a theoretical framework on how to develop students' mathematical reasoning while simultaneously exercise scientific inquiry. Several case studies were designed to test the framework and their findings were summarized. These case studies include detailed instructional supports that were to guide students through the process of merging

[1]Einstein, A. (1914). Principles of theoretical physics, inaugural address before the Prussian Academy of Sciences, 1914. Reprinted in Einstein, A. (1973). *Ideas and opinions* (pp. 221–223). London: Souvenir Press.

mathematical and scientific reasonings in a coherent STEM inquiry. The book can be considered as a resource for STEM education students, researchers, and practitioners seeking to develop transdisciplinary learning experiences.

Structure of the Book

The book consists of two main parts. Part 1 contains six chapters and discusses underpinnings of STEM experiences that lead to a formulation of an integrated theoretical framework. Part 2 comprises four chapters with four STEM projects designated for precalculus and calculus students during which the theoretical framework was put in practice and the outputs discussed.

In Chap. 1, general foundations of integrated learning are discussed, and an argument that STEM students' readiness can be initiated from developing their mathematical reasoning skills using scientific contexts before engaging in engineering designs is posited. Chapter 2 delves more in depth into the current findings on integrated science and mathematics learning and establishes this learning setting as a viable foundation for developing students' STEM competency. STEM learning is strongly supported by representations. Chapter 3 is dedicated to discussing learning effects of using representations in school practice and their effects on STEM practice. While there are various ways of designing and performing STEM activities when representations are enabled, modeling is being used in all of the STEM component disciplines. Chapter 4 zooms into the underlying principles of modeling in biology, physics, chemistry, mathematics, and engineering. It also discusses ways of integrating technology in STEM learning environment. Chapter 5 focuses on synthesizing research findings on using scientific methods in STEM practice. The ultimate goal of this chapter is to seek ways of bridging problem-solving in mathematics with scientific methods. A culminating phase of Part 1 of the book is Chap. 6 that suggests a theoretical framework for merging mathematical reasoning with scientific methods. Intertwining of these methodologies along with applying it in practice is discussed.

Part 2 of the book opens up with Chap. 7 that contains a case study about constructing an exponential model for a bouncing ball and using it to model the law of conservation of energy. Extracting properties of an exponential behavior from that context revealed new knowledge about interpretation of the base of exponential decay that does not parallel with its traditional view. Chapter 8 is about using the idea of function continuity to support constructing a piecewise position function of the simulated motion of a walking man. The fact that position function must be continuous and differentiable is not emphasized in calculus nor physics textbooks. Applying the principle of continuity in real life highlighted the importance of studying the principles and revealed interpretations that do not surface in traditional context-free textbook problems. The concept of function transformations used as a tool to produce new functions based on parent function was explored in Chap. 9. By being situated in a simulated environment of projectile motion, this activity revealed that parent function does not necessarily have to be expressed in its traditional standard form. The simulation also disclosed certain limitations of applying functions transformation in real contexts that do not surface in typical textbook questions. Investigating function rate of change and using it to optimizing area enclosed by a perimeter of a fixed length was the primary objective of the activity included in Chap. 10. While traditional textbook problems ask for unique values that optimize the quantity of interest, this activity offered students the opportunity of constructing functions, exploring their properties and then optimizing the quantity while reflecting on the conducted lab. General conclusions and suggestions for further research conclude each chapter.

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Part II Scientific and Mathematics Reasoning in STEM Practice

Part I
Scientific and Mathematics Reasoning in STEM: Conceptual Framework

Chapter 1
STEM Education: A Platform for Multidisciplinary Learning

Abstract This chapter summarizes general purposes, the learning settings, and the outcomes of using the STEM as a platform for multidisciplinary learning. By encompassing several disciplines, mathematics, physics, chemistry, biology, technology, and engineering, exercising STEM activities posit specific challenges. These challenges are especially visible in high school where students learn contents of STEM subjects in uncorrelated manners. While exercising multidisciplinary STEM activities during extra designed instructional units would be the most efficient, this approach might be problematic to put in practice. Therefore, alternative routes of exercising STEM learning experiences are sought. In this chapter, a framework for an alternative route is suggested and its general theoretical underpinnings discussed. Attention is given to research findings on ways of exercising scientific inquiry and mathematical reasoning in STEM practice. These ideas will also be further discussed in the next chapters.

1.1 STEM Learning Designs

Due to a broad range of aims in educational research and practice, the acronym STEM has multiple definitions. Moore et al. (2014) described STEM as an effort to link some or all the four disciplines of science, technology, engineering, and mathematics into one unit that is based on connections between these and real-word problems. McComas (2014) defined STEM as an interdisciplinary approach to learning that integrates academic concepts with real-world situations. Sanders (2009) described integrated STEM education as proposals that explore teaching and learning between or among any two or more of the STEM subject areas. National Science Foundation defined STEM as the integration of subjects, which include not only the standard disciplines of mathematics, natural sciences, engineering, computer, and information sciences, but also social and behavioral sciences, economics, sociology, and political science (Green 2007).

A large range of STEM interpretations is followed by a high diversity of organizing and delivering these integrated learning experiences to students. These settings vary and one of them, called multidisciplinary, is about applying knowledge

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and skills learned from two or more disciplines to help enhance the learning experience (Vasquez et al. 2013). For example, the idea of periodic functions, from trigonometry, might be integrated with periodical processes studied in biology or physics where students would use real data to construct such functions and learn more about the system behavior. Similarly, the fundamental theorem of calculus might be applied to kinematics to support the idea that accumulation under the graph represents the change of object's position. Such integrated projects can be conducted in math or science classes depending on the content emphasis. In addition to enhancing the learning of the involved subjects, they can provide opportunities for creating new knowledge.

STEM activities can also be classified depending on the form of the final product. For instance; they can lead to formulating a mathematical model of a phenomenon, or be of a form of a theoretical design, or a lead to constructing of an artifact. The nature of the final product depends on the objective of the activity, the time devoted to its completion, and available resources. The form of the final product will support the contents of the involved subjects and amplify the methods of learning. If the final product is an algebraic representation, the effort will evolve around primarily using the attributes of algebraic functions and map them into the observed behavior of the system under investigation. If the final product is an artifact, technical skills will be promoted with applications of algebraic algorithms.

While STEM epistemological framework is rooted in the various disciplines, students' learning can also take different routes; the learning can be teacher- or student centered. STEM activities can be assigned as team projects defined as a process of working collectively to achieve a common objective or as individual projects. Research shows that teamwork produces higher learning effects when compared to individual work because: (a) most engineering designs is done cooperatively, not individually, and technical skills are sometimes equally important as interpersonal skills (Felder et al. 2000); (b) scientists work mostly in groups and less often as isolated investigators, thus similarly, students should gain experience sharing responsibility for learning with each other; (c) cooperative and team learning appears to be the most thoroughly researched instructional methods in all structures of education, (see, e.g., Alters and Nelson 2002). Springer et al. (1999) meta-analyzed STEM learning outcomes and found out that STEM-related cooperative learning promotes greater academic achievement and more favorable attitudes toward learning than traditional students-centered teaching. Thus, communication and teamwork should be prioritized during STEM projects. In such learning, students communicate and discuss thus learn how to convey their arguments to their peers using the language of science, mathematics, or engineering. Team members tend to share knowledge and complement each other's skills which can produce a higher quality of the final projects.

1.2 Learning Outcomes Achieved Using STEM Learning Settings

There are several educational objectives that can be achieved using a multidisciplinary learning environment: (a) promoting awareness of the roles of science, technology, engineering, and mathematics in modern society; (b) enhancing familiarity with at least some of the fundamental concepts from each area; (c) allowing for integrating different teaching methods; (d) promoting active learning (Felder et al. 2000); (e) fostering students' mathematical and scientific reasoning. Among these, using the STEM to promoting active learning is one of the main educational objectives because research shows (Cabrera and Cabrera 2005) that people acquire and retain knowledge and skills more efficiently through practice. While a straight lecturing may succeed at promoting short-term factual recall, active learning promotes long-term retention of information, comprehension, motivation to learn, and subsequent interest in the subject. STEM education also provides multiple opportunities to link scientific inquiry by formulating hypotheses that are proved or disproved through investigations before students engage in the engineering designs to solve problems (Kennedy and Odell 2014). Findings from preliminary studies (Honey et al. 2014) suggested that integration can lead to improved conceptual learning in the disciplines and that the effects differ depending on: (a) the nature of the integration, (b) the outcomes measured, and (c) the students' prior knowledge and experience. Another view by Koedinger et al. (2012) posited that integrated approaches benefit those individuals who already possess some knowledge pertinent to the integrated concepts, as compared to individuals with limited knowledge or less adept at building connections among conceptual structures.

Along with an increasing role of technology, a new STEM learning platform can elicit perspectives on multidisciplinary learning and open the opportunities to extend areas not typically found in subject-specific textbooks.

1.3 Suggested Pathway to Develop Students' STEM Readiness

While STEM is broadly promoted and recognized in education, research does not offer explicit suggestions on what STEM format or what organizational setting maximizes the learning effects (see Barrett et al. 2014). Explicit elaboration on the integrated epistemic goals in STEM education is also rarely found and literature and mostly it refers to either recognizing and applying concepts that have different meanings across disciplines and merging them in one coherent concept or combining practices from two or more STEM disciplines (e.g., scientific experimentation with developing methods of quantifications or scientific experimentation with an engineering design). It is common in STEM practices that one subject takes a dominant role in the learning objectives. As the integration of disciplines is a standard umbrella for exercising STEM practices, the methodology of the integration is still being debated. In their newly developed theoretical framework for STEM education, Kelley and Knowles (2016) suggested applying situated learning to integrate engineering design, scientific inquiry, technological literacy, and mathematical thinking, whereas Vasquez et al. (2013) proposed a continuum of integration through transdisciplinary approaches supported by interconnection and interdependence among the disciplines. Because inquiry-based instruction engages students to think and act like scientists, which is a signature pedagogy in science (Crippen and Archambault 2012) scientific inquiry-based instruction appears as a hallmark of integrated STEM education. Pinar (2004, p. 25) claimed that multidisciplinary curriculum fosters intellectual development and students' capacities for critical thinking. He also contended (p. 25) that "well-designed curriculum should enable students to connect their experiences with academic knowledge." Integration should not only be discipline-wise but also move beyond these boundaries, include students' prior experiences, and provide them with opportunities to construct new knowledge. Mentzer et al. (2014) suggested that teachers should seek opportunities to demonstrate the value of mathematical modeling and encourage students to think about relationships and functions as ways of understanding the world around us. While research on the impact of integrated STEM experiences on students' achievement, subject–domain knowledge, problem-solving ability, and the ability to make connections between disciplines is not extensive, concerns related to both the design of studies and the reporting of results hamper a need to make explicit claims about what areas of students' learning is affected the most.

While all types of discipline or methods of integration generate learning, the most potent are these that include scientific methods (Davison et al. 1995). Therefore, this area will be further discussed and explored in this book. How to integrate abstract math structures with hypotheses formulation that are typical for scientific investigation? Alternatively, how to design the process of the derived model verification that will reflect on both scientific phenomena embedded in the investigation and the algebraic structure? While it is apparent that through modeling processes, students will have the opportunities to improve their mathematics and scientific reasoning skills, the underlying question is how to integrate inductively organized learning experiences in sciences with a traditionally structured mathematics learning. Is there a common area for such integration? Alternatively, how to organize activities that would enhance a pathway suggested in Fig. 1.1?

To develop a more concise theoretical framework for multidisciplinary learning, currently used modeling in all STEM component disciplines will be analyzed (see Chap. 4). Following this analysis, a survey of research on using scientific contexts in mathematics classes will also be discussed, and it will be summarized in Chap. 5. English (2016, p.1) claimed that "We still need more studies on how student's learning outcomes arise not only from different forms of STEM integration but also from the particular disciplines that are being integrated." The book can be considered a response to this call.

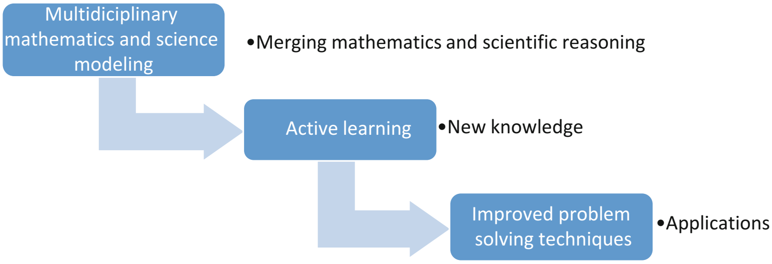

Fig. 1.1 Proposed epistemology of multidisciplinary STEM modeling

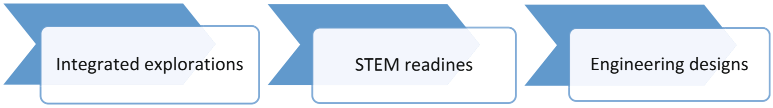

Fig. 1.2 Pathway to develop students' STEM readiness

1.4 Enhancing Scientific Route in STEM Learning Settings

Students' successes in science and engineering significantly depend on their skills of mathematical modeling where mathematics with its tools is the language to derive quantitative solutions (Dym 2004). This relationship suggests that emphasizing modeling in high school mathematics classes might benefit potential engineering students and assure their college readiness. A broad range of disciplines and learning methods included in STEM activities that extend from learning scientific principles and their techniques of quantification to designing artifacts makes it challenging to organize. This challenge perhaps accounts for the earlier discussed difficulties that are faced by the STEM education community regarding organizing STEM activities and evaluating the learning effects. In the light of that, it is prudent to try to develop integrated learning experiences that will encompass a smaller range of objectives, but that will simultaneously get students ready to immerse in more complex engineering designs at college levels. A proposed pathway to accomplish that goal is illustrated in Fig. 1.2.

The primary objectives of integrated explorations designed this way are developing students' mathematical reasoning using scientific contexts and developing students' ability to learn about scientific phenomena using algebraic structures. For the purpose of the book, merging both this reasoning will be called STEM reasoning. The main goal of such sequencing is to provide experiences to develop students' subject knowledge and STEM reasoning prior building artifacts. In such conducts, performing experiments will inform the learners about the function and performance of potential design solutions before a prototype involving such investigations is constructed. Initiating STEM engagement from merging science and mathematics was also advocated by Kennedy and Odell (2014) who postulated that STEM education must provide opportunities to have students experience scientific inquiry before they engage in the engineering designs to solve problems. These recommendations suggest that STEM designs can be exercised in two internally cohesive stages: the first that is to have students experience merging quantifying methods of mathematics with scientific inquiry and the second that will take these skills to the next level and have students apply this reasoning in engineering designs. In the next five chapters, research findings supporting such pathway within STEM learning will be presented and discussed.

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Chapter 2
Integrating Mathematics and Science Within STEM Paradigm

Abstract This chapter discusses research findings on integrating mathematics and science within STEM platform. It summarizes the premises that benefit the learners as seen from the perspective of mathematics and science content knowledge and highlights the areas that need more attention. In the traditional curriculum, science provides the contexts and mathematics and offers the tools to quantify the contexts. While students do use the tools of mathematics to solve problems in science and use scientific contexts in mathematics, research shows that the methods applied are often disconnected. STEM activities can be seen as offering opportunities to create a new, unique knowledge rooted in merging these two disciplines into an integrated inquiry. In the attempt to merge these two learning disciplines in such a way, this chapter also provides an analysis of the primary phases of scientific investigation and its possible fit to STEM mathematics activities. A draft of the general theoretical framework on merging scientific inquiry with mathematical reasoning and its relation to STEM competencies is also brought to the reader's attention in this chapter.

2.1 Mathematical Reasoning and Scientific Inquiry

The most frequently researched STEM education pairing is that of mathematics and science (Marginson et al. 2013). Mathematics provides a computational system and helps conveniently encode a rule (Bing and Redish 2008). As a teaching and learning real-world application problems is difficult in mathematics classes (Berry and Nyman 2002), developing students' mathematical reasoning skills by formulating mathematical constructs using STEM contexts appears as a strong opportunity that is not fully explored. An aspect that is not often being undertaken in mathematics classes is coupling algebraic representations with natural phenomena to provide opportunities for enhancing mathematical reasoning. For example, if students find how the speed of a cart behaves over a definite, measurable distance, they can find an algebraic function that models the cart's position and use it to compute the position of the cart at any time beyond the one utilized in the lab.

A high range of mathematical apparatus studied by students allows for describing phenomenon from multiple angles. For example, it allows for: (a) quantifying the

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outputs of a deductive inquiry, e.g., to find unique solutions to problems; (b) formulating general mathematical representations as a result of applying inductive inquiry; (c) representing the data in various representations, graphs, table of values, symbolic; (d) predicting system behavior based on the properties of a corresponding algebraic representation. STEM environments that allow the objectives to put in practice might not only extend students' views of applications of mathematics but also serve as catalysts to spark their interest in mathematical modeling that is essential in engineering courses (Sacks and Barak 2009). While the methods of teaching science can be introduced along with the development of the skills of mathematical modeling, some of the earlier developed integrated methodologies suggested that concepts of mathematics can be contextualized using constructivist theory and science discovery (Davison et al. 1995). These methods assumed that the learner is provided with opportunities to build upon prior knowledge, respond to the new learning environment, and construct knowledge based on these experiences. DiSessa and Sherin (2000) posited that new forms of mathematical expressions supported by modern technology, for instance, by computational media and simulations can generate new ways of helping with applying constructivist theory and scientific discovery. Modern technology in forms of simulated scientific experiments can also make the process of merging mathematics and science reasoning accessible in mathematics classes and make it available to broader populations of students.

2.2 Mathematics as a Tool for Quantifying Scientific Phenomena

Poincare stated that "all laws are derived from experience, and to report them, a special language [. . .] of mathematics is needed" Murzi (2005, p. 67). Mathematics provides scientists with the tools to formulate the laws of nature into a concise and symbolic language. According to Ernest (2010, p. 4). "The concepts of mathematics are derived from direct experience of the physical world, from the generalization and reflective abstraction of previously constructed concepts, by negotiating meanings with others during the discourse, or by some combination of these means." Science provides mathematics with contexts to investigate and to model. However, physical phenomena cannot be completely understood only by mathematical formulas and equations, and in parallel mathematical representations standing alone do not guarantee that integrated learning will be nurtured.

Research shows that STEM activities can be productive if they involve students in generating or refining mathematical representations of the systems given either in static or dynamic forms. Honey et al. (2014) suggested that there needs to be an explicit focus on the mathematics' concepts and processes that arise during the investigations. Without a focus on mathematical methods, the promotion of problem-based STEM tasks might run the risk of sidelining mathematical reasoning

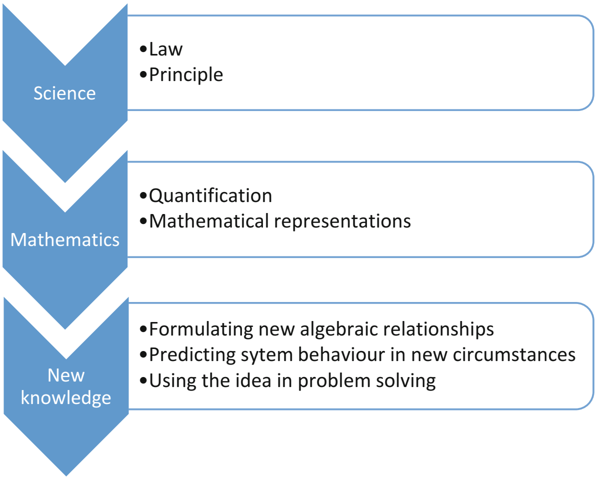


Fig. 2.1 Merging the methods of science and mathematics

into minor roles, e.g., routine algorithmic procedures and graphs sketching. According to National Research Council (2013, p. 5), connecting ideas across disciplines is challenging when students have little or no understanding of the relevant thoughts. The challenge appears to be higher considering that students do not always naturally use their disciplinary knowledge in integrated contexts. Although presenting students a STEM problem without guiding them through the stages of merging different disciplines to solve the problem might result in finding the solution, this approach most likely will not improve the quality of their reasoning skills. Thus, more careful planning is needed. A diagrammatic representation of a sequence of methods that are set up to nurture the development of integrated reasoning skills is suggested in Fig. 2.1.

Integrated mathematics—science learning experiences are initiated from observing a phenomenon. The next step in the process is identifying variables and classifying them as independent and dependent. Taking data, deciding about algebraic representation and then using the representation to reflect back on the system behavior usually concludes the process. By formulating an algebraic representation of the phenomenon, students are given opportunities to contextualize function's attributes such as monotonicity, concavity, the rate of change, domain, range, maximum or minimum values, asymptotes, limits, and so forth. They have a chance to develop the understanding of nature through a concise language of mathematics. By interpreting system behaviors using this language, they will realize the importance of learning tools of mathematics and create or discover new knowledge from the multidisciplinary STEM experience. Kelly (1989, p. 31) claimed that "to acquire knowledge is to have students experience, observe, and form hypothesis." It is anticipated that such created STEM environments will help students not only with appreciating mathematics but also with an understanding of science, where applying the process of inquiry is the primary method of knowledge acquisition. Integrating scientific inquiry with mathematical representations might be a bridge to solidifying the process of creating mathematical models, solving and using the models to exercise mathematical reasoning. Studies (Honey et al. 2014) suggested that the integration of mathematics and science can also be supported by engaging students in the invention and revision of mathematical representations of natural systems because concepts make sense not as isolated facts but as elements of integrated structures of knowledge. Guiding the learners through identifying the pieces of information that are crucial in the process of knowledge integration and yet leaving a room for their judgments and inputs requires a careful analysis of the project' goals and contents of the involved disciplines. A survey of the field of empirical research on using scientific methods in the STEM that will shed more light into current research is synthesized in Chap. 5.

2.3 Merging Mathematical Reasoning with Scientific Contexts

Although educational bodies support STEM, the nature of how to integrate all the disciplines is still being debated. Many have voiced concern that mathematics is underrepresented in the STEM paradigm (e.g., English and King 2015). While no one questions the dominant role of scientific methods in the STEM, Hämäläinen et al. (2014) posited that the role of abstract mathematical concepts could increase if those concepts are considered as a process of giving mathematical structure to theoretical knowledge and empirical observations. This idea is supported by the notion that mathematics can be perceived as a human conceptual construction of embodied concepts (Lakoff and Núñez 2000). It is believed that by inducing mathematical concepts to scientific explorations, such learning settings can also serve as a means of promoting social construction of knowledge as defined by Buendía and Cordero (2005). While in mathematics, the notion of a generic competency often relates to problem-solving, the justification for the current position of mathematics in the curriculum is still around procedural competency (Marginson et al. 2013). In such settings, the role of students is reduced to plugging in values and evaluating expressions, often with the help of a calculator. The limited use of the tools of mathematics to support scientific practice is reflected in low students' competencies in problem-solving. Research (Bonotto 2013) showed that even when the problems and methods encountered in class are similar to the real-world situations, students have difficulties in associating their analytical thinking with their problem-solving techniques. This difficulty might illustrate a gap between how students perceive the concepts of mathematics and the applicability of these concepts to solve real-world problems. There can be many reasons for this gap of skills and pinpointing some based on the gathered literature might be premature. It is hypothesized that this deficiency is attributed to a limited students' exposure to actual experiences because students learn and retain knowledge better by being actively involved in the processes of linking scientific explorations with applications of algebraic functions.

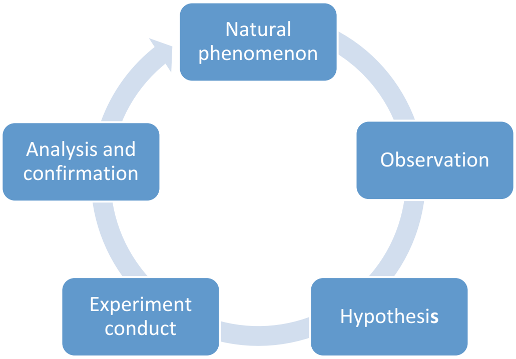
Competency in problem-solving is also one of the central concerns of physics education. For instance, Redish (2017) and Sokolowski (2017) argued that the role of mathematics in constructing quantitative descriptions of scientific phenomena in science school practice is not highlighted enough. Several findings from physics research community describe students' problem-solving strategies as manipulation of formulas by rote. Such use of the tools of mathematics has little to do with applying mathematical reasoning because the reasoning does not make students connect the attributes of algebraic functions with what they observe. It is seen that STEM has a great potential for providing students with opportunities to improve these skills and for rebalancing the minimized role of mathematics in science. STEM can also be used to show students that mathematics should no longer be seen as a discipline studied and applied for mathematics' sake only, but because it helps make sense out of some part of the world that they study in other academic courses. Through the integration process, science and mathematics exchange points of view outside of the paradigm of the scientific or logical-analytical mathematical methods. Before scientific facts can be integrated, they need to be organized into general concepts according to their specific attributes. Thus, learning science enables the skills of classification and categorization of data that translates to applying specific algebraic tools to express the data symbolically. Learning mathematics, on the other hand, should entail using scientific contexts to exercise mathematical reasoning in new more sophisticated dimensions. Integrating scientific inquiry with mathematical reasoning can serve as a bridge to enhancing the process of eliciting mathematical models and to advancing the use of mathematics in other courses.

2.4 Scientific Methods and Inquiry in STEM Learning Settings

Science is the investigation of natural phenomena using scientific methods (Windschitl et al. 2008). Scientific methods include several interconnected phases that are: the careful observation of natural phenomena, the hypothesis formulation, the conducting of one or more congruent experiments to test the hypothesis, and the drawing of a conclusion that confirms, modifies, or refutes the hypothesis. General phases of the scientific process along with the direction of their progression are illustrated in Fig. 2.2.

The cycle is initiated by defying natural phenomena of interest. The investigator immerses then in its stages following the details of the lab-specific procedures. In completing the cycle, the investigator converts the *natural phenomena* to a different, more sophisticated representation, often expressed in a concise language of mathematics. To have the learners succeed in these processes, not only the skills of applying scientific methods are required, but also adequate mathematical skills and

Fig. 2.2 General phases of scientific methods



students' ability to convert the lab prompts into semiotics that can be used to express the lab process in a different representation. In this regard, scientific reasoning is related to cognitive abilities such as critical thinking. Scientific reasoning can be developed through training and can have a long-term impact on student academic achievement. The STEM community considers that these skills are as crucial for students to learn as the STEM content knowledge (Honey et al. 2014).

Scientific methods should be distinguished from the aim of products of science, such as knowledge, prediction, or control. Methods can be perceived as how the goal of the undertaking is achieved. Methods can include specific laboratory techniques, such as taking specific measurements using more sophisticated devices such as probes, or photogates and mathematical apparatus including the use of technology (e.g., the techniques of computing the coefficient of determination or standard deviation) or other specialized software or programs.

An inquiry approach to instruction requires teachers to encourage and model the skills of scientific inquiry, as well as the curiosity, openness to new ideas, and skepticism that characterizes science (National Research Council 2013). Scientific inquiry trains students to ask questions, hypothesize, carry out investigations, and formulate inferences. It guides students to think and act like real scientists. "To engage in authentic and productive inquiry, students must come to understand inquiry not as the accumulation of objective facts but as an enterprise that advances through the coordination of evidence with evolving theories constructed by human knowers" (Kuhn and Pease 2008, p. 513).

There are three main types of reasoning used to formulate new knowledge: inductive, deductive, and abductive. In science, mathematics, and engineering, inductive and deductive reasoning are mainly used (Prince and Felder 2006). The inductive reasoning could occur in three progressively different avenues: *structured inquiry, guided inquiry*, and *open inquiry* (Staver and Bay 1987). Most common inductive inquiry in school practice is a *structured inquiry* where students are given a problem or equipment and some auxiliary information that guides them through the solution process.

While a deductive inquiry is a process of reasoning from specific observations to reaching a general conclusion (see Fig. 2.3), an inductive inquiry denotes the process

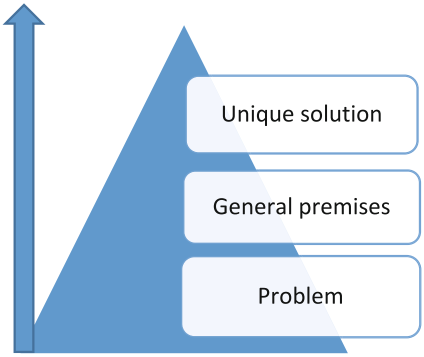


Fig. 2.3 Process of an inductive reasoning

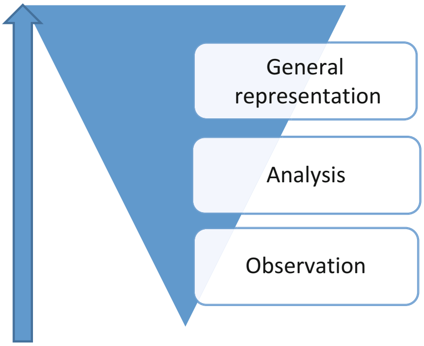


Fig. 2.4 Process of a deductive reasoning

of reasoning from a set of general premises to achieving a logically valid conclusion (see Fig. 2.4).

In sum, deductive thinking draws out conclusions, whereas inductive thinking adds information (Klauer 1989). Abductive reasoning, also called abduction, is an inference representing the best possible explanation (Thagard and Shelley 1997). It is a type of inference often expressed in a freely defined form that can be found in everyday events and scientific reasoning. Since the exact form, as well as the normative status of abduction, is still a matter of controversy, this type of reasoning is not considered to support the reasoning behind the proposed STEM activities.

The mathematics education community has taken the position that observation, experiment, discovery, and conjecture are as much a part of the practice of teaching and learning mathematics as of any natural science (National Research Council [NRC] 2013). STEM activities that offer explorations and learning contexts during which students apply theorems to create new knowledge through experimentation present an excellent platform for following this recommendation. It is believed that using scientific methods in mathematical activities will help students perceive mathematical concepts as tools to learn the laws of nature and to construct, test, and validate engineering designs. A far-reaching goal of this enterprise is to produce

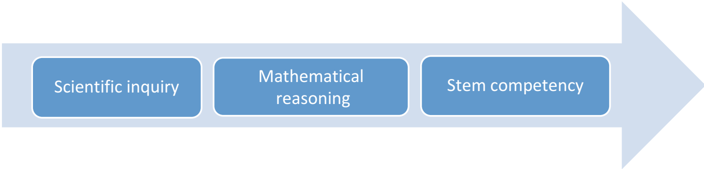


Fig. 2.5 Pathway to develop competency in STEM education

competent STEM modelers that will be ready to take the risk to develop new technological devices. A pictorial summary of this thesis is illustrated in Fig. 2.5.

The theme of the book is to support the notion that STEM activities have the potential to serve as a means of fostering students' mathematical reasoning and consequently improving their scientific modeling skills. The next chapter provides more detailed background about why real contexts of STEM activities help with knowledge accumulation and retention.

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Chapter 3
Teaching and Learning Representations in STEM

Abstract Context in the STEM is a critical factor in learning. Context can be delivered in various ways depending on the form of the final product. Research shows that representations are very effective in conveying knowledge because they help learners visualize abstract ideas and diversify the forms of information. This chapter discusses the effects of representations on learning and attempts to answer a question why representations support knowledge acquisition and retention. Representations can function in two primary capacities: as provided by the instructor or produced by the learners. Being able to gain understanding using representations and constructing representations is one of the most critical factors in supporting knowledge retention. What are the features of well-designed representations and how they affect knowledge processing are other questions that this chapter also attempts to answer.

3.1 Representations as a Means of Supporting Learning

When applied to STEM, representations can function in various capacities and be derived in a number of ways. Representations can be dynamic and interactive; they can serve as resources for reasoning, predicting, hypothesizing, visualizing, testing, and confirming students' prior experiences. According to National Research Council (2007), representations can also encompass clarification of problems, deduction of consequences, and development of appropriate tools. In learning mathematics and science, developed models also called the products of representations, have a high impact on students' further engagement and success in these subjects. Exploring these capacities during STEM activities can benefit not only STEM disposition but also their success in other subjects outside of the STEM domain.

STEM learning environments in which the students observe real experiments formulate their graphical and symbolic representations or build artifacts provide excellent learning opportunities for developing knowledge based on a concise language of representations. In this pipeline, representations encompass physically embodied, observable configurations—such as pictures, concrete materials, tables, equations, diagrams, along with various forms of schemata and drawings of one-,

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two-, or three-dimensional figures (Kaput 1989). Representations can also include algebraic functions that learners formulate as a product of investigating dynamic or static systems. All of these embodiments are also called external representations, and they can constitute learners' tasks or be provided in the form of drawings, artifacts, or interactive simulations. Representations promote conceptual innovation of depicting natural phenomena by embedding them in inscriptions that reduce some aspects of the target and amplify others in a way that allows the learners to focus on the meaningful aspects (Lehrer 2009). Representations can serve as a means to achieve a final product or a way of setting forth a situation or formulating a problem so that efficient and acceptable solution to a problem can be found (Dym and Brown 2012). In sum, the purpose of using representations is to synthesize aspects of a system while simultaneously highlighting its essential elements. A significant insight from research on cognition and learning is that the organization of knowledge, thus the ability to make connections between concepts and representations is a key to the development of expertise in a domain (Ainsworth 2006).

While in engineering the term representation means to draw the plan for and to create, execute, or construct an artifact according to the plan, in this book, the term will take a broader meaning. By working on activities that will require integration of knowledge, students will practice the skills of constructing representations. The representations will integrate mathematical concepts with scientific phenomena. Students will also practice retrieving integrated information from provided representations or transferring the representation into new knowledge. Research shows (Cook 2006) that representations help with understanding because the learner can easily identify meaningful pieces of information and link the information with his/her own prior experiences. By being described this way, the knowledge retention is more efficient, and it provides a mechanism for a faster recalling. However, the key to develop expertise in the scientific and algebraic inquiry is the ability to make *connections* between concepts and representations. Thus, representations in school practice should be perceived as a bridge between abstract theoretical knowledge and learners' prior experiences. By converting experiment's cause and effect into a graphical representation, students visualize the realm that enables them to conceptualize the mutual dependence of the cause and effect. Consequently, acquiring such ability should help students to convert the experiment's causes and effects into a specific algebraic representation.

The matrix (see Fig. 3.1) summarizes some advantages of using representations to support learning.

STEM learning environments in which students are engaged in observing real experiments, formulating graphical or symbolic representations or building artifacts provide excellent learning opportunities for developing the skills of retrieving essential pieces of knowledge from representations. Representations in school practice can be perceived as bridges between abstract theoretical knowledge and the world outside of the classroom.

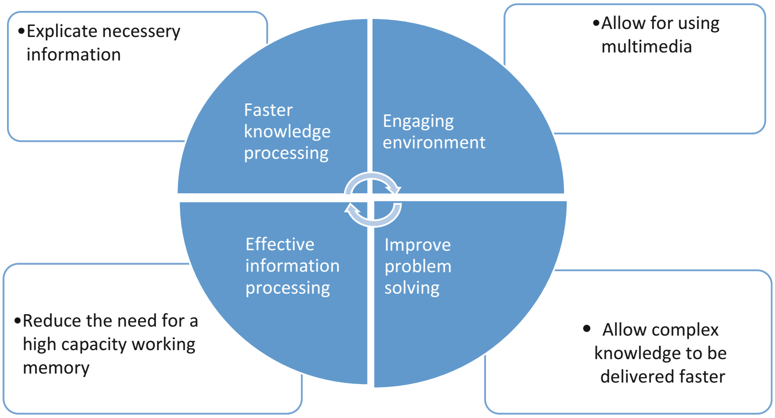

Fig. 3.1 The effects of representations on learning

3.2 Representations in Mathematics and Science

Identifying representations as a means of learning corresponds with a modern view of mathematical and science education, which calls for making connections between abstract, graphical, symbolic, and verbal descriptions of relationships and their artifacts (Gilbert 2005). Prain and Waldrip (2006) pointed out that an empirical-mathematical modeling approach to teaching science is beneficial for students when it is accompanied by explicit and integrated attention to the nature of science and students' learning strategies.

While learning or constructing mathematical or science knowledge involves not only manipulating on symbols but also identifying relationships and interpreting the relationships, graphical representations, especially their dynamic embodiments have a high potential to help students with learning these processes. As Eisenberg and Dreyfus (1991, February) noted, students might end up with an incorrect solution if their algebraic skills are not strong even though their reasoning might be correct. If the learner possesses the skills of graphically expressing the problem or support its solution process, the representation might serve as a backup or a means of verifying the algebraic solution. By being versed in constructing and applying representations, the learners "acquire a set of tools that significantly expand their capacity to model and interpret physical, social, and mathematical phenomena" (NCTM 2000, p. 4). Modern technology provides multiple advantages of exploring these features. Producing representations also plays a vital role in deriving new theories; in fact, the majority of scientists made their discoveries by carefully selecting and analyzing representations or by inventing new once (Cheng 1999). Possessing the ability to convert observed phenomena into graphical formats such as diagrams, graphs, or algebraic functions can help underpin its underlying principle. This skill consequently should help students understand the phenomena properties and provide a basis for using this understanding in further inquiries or problem-solving.

Despite extensive research, it is still debated how to make the transition from abstraction to representation in the way that would be most accessible to students. The proposed pairing of scientific and algebraic concepts (see Chaps. 7–10) can be seen as one of the ways. The other question that arose from the literature analysis is if representations used in science are coherent with representations used in mathematics or if representations used in mathematics support accurately scientific concepts? Alternatively, what is the extent to which there should be a mutual coherence of representations so that the learner can use the knowledge of more than one subject to explore concept using the representation? For example, a preliminary survey of college-level physics resources showed that graphs of functions used in physics not always satisfy the conditions for being functions from a mathematics point of view (Sokolowski 2017). Can STEM activities serve as a bridge of unifying these representations and provide a coherent view? While developing students' skills of converting between representations has proven to be beneficial for students' learning, STEM environments can serve as an excellent basis for enhancing this learning competency.

3.3 Active Learning and Representations

The strength of the knowledge delivered by representations is supported by the constructivist learning theory (see Dewey 1997). According to that theory, knowledge is not passively accumulated, but instead, it is the result of active cognizing by the individual. STEM by promoting active learning when students are provided with tangible experiences reflects on that theory. The learning effects of the constructivist theory have also an impact on mathematics education. Hoeffler and Leutner (2007) contended that learning is more actively accumulated when representations are utilized to convey the meaning of mathematical ideas. Active learning has consistently been shown to be superior for promoting long-term retention of information, comprehension, problem-solving skills, and interest in the subject (Felder et al. 2000).

STEM projects provide multiple opportunities to generate active learning. Active learning will be promoted in the practice section of the book by generating environments where students will have a chance to set up, observe, and interact with experiments' outputs and reflect on their experiences.

3.4 Dual Channel of Knowledge Processing

Clark and Mayer (2016) claimed that knowledge accumulation is supported by the following principles of learning: (a) dual channel of information processing—people have separate channels for processing visual/pictorial material and auditory/verbal material; (b) awareness of limited memory capacity—people can actively handle only a few pieces of information in each channel at one time; and (c) active processing of

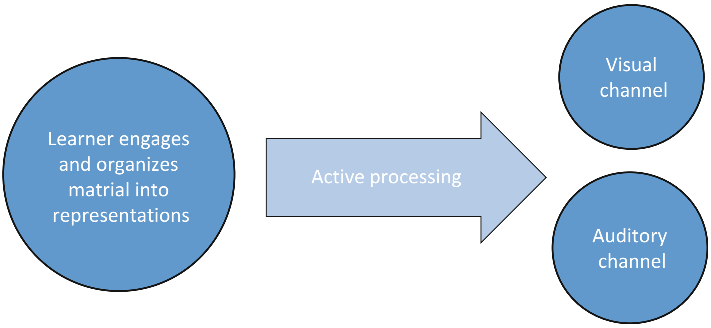

Fig. 3.2 Duality of information processing (inspired by Clark and Mayer (2016))

information—learning occurs when people engage in appropriate cognitive processing such as attending to relevant material and organizing the material into coherent structures. The principles of learning reflect on how knowledge is stored and retrieved from learners' memory. By carefully balanced activities, during which the learner is prompted to justify, for example, the sound frequency and simultaneously observe the wave, the information about the pitch of the sound is diverted into two learning channels, *visual* and *auditory*, that allow for faster and more efficient processing. While *visual* information includes pictures, diagrams, charts, plots, animations, and so forth, and *auditory* information includes spoken words and other sounds. The medium of information that needs more elaboration is *written prose*. Cognitive scientists have established that human brains convert written words into their spoken equivalents and process them in the same way spoken words are processed (Felder and Henriques 1995). Thus, written and read words are classified as *verbal representations* processed by an auditory channel (Fig. 3.2).

Several researchers proved that student learning improves when single sources of information are visually integrated so they can be processed together in a single image (Moreno and Mayer 1999). Embedding meaningful, integrated, and conveyable to students' representations during STEM activities in the tasks of creating representations or retrieving information might benefit students' general academic disposition.

3.5 Human Memory and Its Capacity

The mechanism of converting information into graphical representations plays a vital role in how the information is stored in memory. Hollingworth and Henderson (2002) showed that memory created by graphical representations is retained longer than the memory of spoken words. Paas et al. (2003) stated that two structures affect the rate of information processing: working memory and long-term memory. Human working memory has a limited capacity, whereas long-term memory has an unlimited capacity (Kintsch 1998). For the information to be stored in a learner's long-term memory, it needs to be processed initially through its working stage. Thus, if students are given a dose of complex information, they might feel overwhelmed, which can result in the information not being fully processed at the first stage, thus not understood and learned. In such case, the information will be blocked from being accumulated in the learners' long-term memory. The primary goal of using representations is to convert, and thus reduce the information to a concise visual form and to transmit it to the students' visual channel. STEM learning environments where students engage in projects that have them merge observable experiments with graphical and symbolic representations provide great opportunities for helping attend and process the knowledge through a visual channel. Research (Kintsch 1998) shows that knowledge treated by a visual channel has a higher probability of being stored in the long-term memory.

Effective teaching is about formulating and delivering new knowledge in a way that is encoded in students' long-term memories. Cognitive research (Ainsworth 2006) informs that information that is not related to learner's existing knowledge is likely not to be retained. Moreover, to recall and use the knowledge again cues are required. Linking the new material to prior structures provides a natural set of cues (Felder et al. 2000). While converting knowledge to a different representation, its cognitive value cannot be reduced, but only converted to different and more accessible forms. This process, per human cognitive architecture (Shepard 1967), reduces the need for high capacity working memory and allows the information to be accumulated in the learner's long-term memory. The virtue of using dynamic or static representations that depict realm lies in being able to express or convert knowledge in convenient graphical embodiments supported by verbal elaborations rather than vice versa. Such knowledge presentation creates appealing conditions for being accumulated: longer retained and accessible for further retrieval, e.g., during assessments. Disciplines like engineering, mathematics, and science that are predominated by real applications are particularly susceptible to be supported by representations.

3.6 The Effects of Internal Representations on Knowledge Acquisition

Mediated by the level of entry into learners' memory system, Kaput (1989) categorized representations as external, discussed (in Sects. 3.1 and 3.4) and internal that constitute the accumulated knowledge. Both types of representations are interrelated in the sense that the meaning of internal representations stored in a learner's long-term memory strongly depends on the student's perception of the external counterpart. In the process of accumulating new learning experiences, external representations are converted into internal images that encompass mental images corresponding to internal formulations of what human beings perceive through their senses. Internal representations cannot be directly observed. They are defined as the learning experiences stored in learner's long-term memory (Hochreiter and

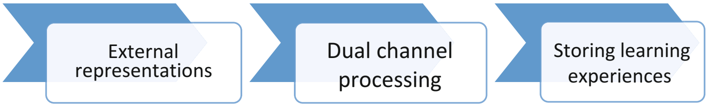

Fig. 3.3 Stages of storing learning experiences in long-term memory

Schmidhuber 1997). Internal representations are formulated based on one's interaction with the environment and are altered throughout a life span. In the process of learning, internal representations are prompted by external representations. Being able to formulate concepts' internal representations plays an essential role in communicating knowledge. Hiebert and Carpenter (1992) maintained that there exists a strong relationship between external and internal representations created by learners and that the extent of linking these representations determines *understanding*. It has been proven that the most effective transitioning of knowledge from short-term memory to learner's long-term memory is through experiencing *external visual* representations. Perkins and Unger (1994) stated that "mental maps or mental models or other sorts of mental representations mediate with what we would call understanding performances" (p. 4). Viewed through these prisms, the internal representations assert knowledge as a body dependent on a learner's own experiences. The process of accumulating these experiences is illustrated in Fig. 3.3.

Enabling the processes of creating representations by supplying engaging, yet intellectually stimulating environments deems to be one of the most fundamental factors in nurturing successful learning. Being situated in realistic settings, STEM provides opportunities for exercising not only formulating representations but also for verifying their effectiveness and the adherence to reality. "Design of integrated experiences must balance the richness of integration and real-world contexts against the constraints of the cognitive demands of processing information that is separated in time, in space, or across disciplines and types of representation" (Honey et al. 2014, p. 98). It is vital that the prompts created by learning experiences during STEM activities be diverted into visual and auditory channels. By allowing students to discuss their experiences and inferences while observing the behaviors of their projects or experiments learning will be well nurtured. Such learning setting will reduce the need for a high capacity working memory resulting in knowledge being easier accessed by the learner. I will discuss learning settings of individual STEM disciplines in Chap. 4. Their standard features will be extracted, and an attempt to integrate them in a coherent STEM inquiry will be made.

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Chapter 4
Modeling in STEM

Abstract Knowledge in sciences is traditionally derived inductively, whereas in mathematics and engineering it is derived deductively. One of the purposes of STEM projects is to blend science with mathematics and engineering into a coherent learning experience. While this book is an attempt to develop and put in practice a theoretical framework for exercising multidisciplinary STEM learning experiences, a need to discuss a strategy that would link all learning methods of the component STEM subjects emerged. Research shows that modeling is the most common approach exercised in all of these disciplines. However, the phases of modeling in mathematics might not parallel with the phases of modeling in science. To design a learning environment that would support multidisciplinary learning, a need for a modeling cycle that would integrate the features of all types of STEM modeling appeared as a necessary step before the multidisciplinary projects could be designed. The purpose of this chapter is to synthesize the characteristic features of currently applied modeling cycles in each of the STEM disciplines and select these features that can support STEM learning objectives exemplified in this book.

4.1 Modeling and Construction of New Knowledge

Although educational bodies support STEM, a high diversity of modeling and learning methodologies can be one of the reasons why integrating the contexts of STEM disciplines and evaluating learning effects is still debatable. While active learning and constructivist theory serve as guidance for efficient learning, modeling provides a framework for the activities design. However, in the attempt to merge all modeling approaches in one modeling process, a reference to current literature on modeling was needed.

The idea of modeling is inherent in the scientific methods, and it is a central feature of engineering (Felder and Brent 2003). Modeling is also a framework for organizing a learning environment. Some studies (e.g., Malone 2008) find that modeling instructions promote expert problem-solving behavior in students. By being applied to various disciplines, sciences, mathematics, and engineering, modeling is defined in some ways. Odenbaugh (2005) defined it as a collection of cognitive

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strategies that can be used to pursue aims of scientific inquiry. In this view, the product of modeling takes a form of new knowledge, yet grounded in the scientific methods. Wilkinson (2011) defined modeling as an attempt to describe, in a precise way, an understanding of elements of a system of interest, their states, and their interactions with other items. Modeling is also perceived as a process of *approximation* of reality. Cobelli and Carson (2001) described modeling as a process that is initiated by isolating a system and applying methods that would enable to learn its structure. Such defined system represents a part of *reality* under investigation. By applying suitable, for a specific phenomenon, modeling strategy, a model is formulated. The model is to reflect on the system parameters that were of interest to the researcher.

Models, the products of modeling, can be generated by the use of *analogy*. The analogy can be made with the *target* of modeling or by a partial comparison with a *source* (Clement 2008). In this view, it is helpful for the modelers to develop a network of sources, for example, mathematical representations, for comparing target of modeling with the sources. It is necessary that the learners acquire a clear understanding of various algebraic structures along with their specific characteristics prior engaging in modeling activities. The algebraic structures, as mentioned earlier, will serve as *source* models for *target* STEM modeling. Models, similar to representations, can take different forms: conceptual, mental, verbal, physical, statistical, logical, graphical, and so forth. According to National Research Council (2012), modeling is more commonly understood as an integral component of authentic scientific inquiry. Therefore, model-based curricula have been gaining more attention in current education reform.

The following subsections summarize the critical stages of the modeling cycles used in STEM this in biology, chemistry, physics, mathematics, and engineering. Research shows that in the STEM education the area of merging science methods with mathematical reasoning is underrepresented. Thus, more attention will be given on how modeling in mathematics can support other STEM disciples modeling cycles. A synthesis of all key modeling stages will serve as a departing point to developing a general framework that is presented in Chap. 5.

4.2 Research and Modeling in Biology

Research in biology education endured a significant change recently due to the integration of computational technologies and methods from physical sciences, mathematics, computational sciences, and engineering (National Research Council 2012). The effects of using methods of quantification in biology constitute a substantial part of biology education (Steen 2005). This change implies for instance that biology students should be able to demonstrate quantitative numeracy and familiarity with the language and tools of mathematics, make inferences about

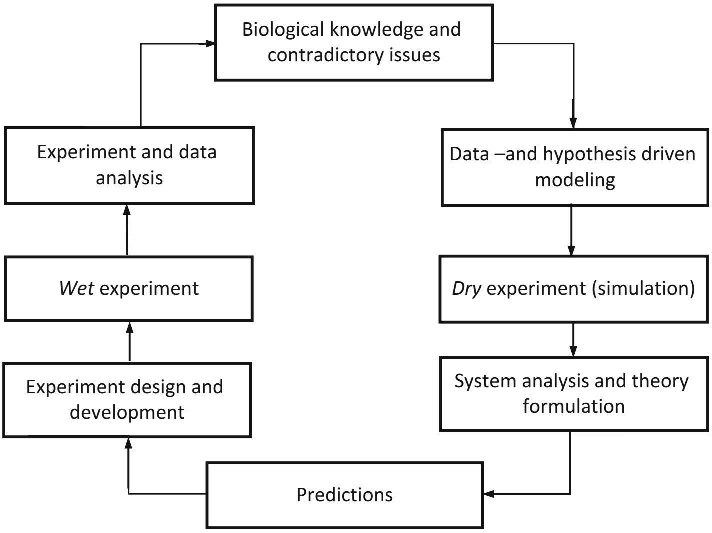

Fig. 4.1 Modeling framework in biology inspired by MoKitano (2002)

natural phenomena using mathematical models, and make a distinction between cause and effect in problem-solving (Mayes and Myers 2014). The diagram (Fig. 4.1) illustrates a standard modeling cycle in biology developed by MoKitano (2002). The process is initiated by a selection of a problem, and it is followed by formulating a hypothesis and creating a model representing the phenomenon. The so-called *Dry experiment* often supported by a digitally simulated environment supplies means for analysis and verification of quantified nature of the formulated model. If the anticipated model fails to represent the simulated environment adequately, it is rejected and resigned. Models that pass the tests become subject to a further more detailed analysis, and eventually, they are deployed to formulate a set of predictions used to formulate a *Wet experiment*. These models that prove to produce outputs consistent with real (*Wet*) experiments are ready to be applied in real-life situations. Advancing in computational science makes the modeling cycle a promising tool in research and biology education.

The biology education research community faces specific challenges that impede the advancement of modeling biological processes. Some of these challenges are (a) a lack of correlation between biology and mathematics curricula that prevents using more advanced mathematical apparatus in biology classes and (b) a deficiency of appropriate examples involving biology contexts in mathematics classes (Gilbert and Boulter 2012a, b). It is seen that STEM platform can serve as a means to help reduce these challenges.

4.3 Modeling in Chemistry

Chemistry is "the scientific study of the properties of the composition, the structure of matter, the changes and accompanying energy" (McGraw-Hill Dictionary 2003, p. 376). Chemical ideas have been visual, mathematically or verbally modeled, and recently computational models and modeling have a firm establishment in chemical research and education (Erduran 2001). Models in chemistry are framed by bringing history and philosophy of chemistry together. However, the formulation and testing of these models takes place in a chemistry laboratory (Justi and Gilbert 2002). Unlike in physics where the tendency of modeling is mathematization, chemistry modeling relies on classification schemes which focus on explaining the qualitative aspects of matter (Moyer et al. 2007).

All modeling activities in chemistry are undertaken with a *purpose,* whether it is to describe the behavior of a phenomenon, to outline the reasons for the causes and effects of that behavior, or to predict how it will behave under other circumstances (Gilbert 2002). An example of modeling process in chemistry is illustrated in Fig. 4.2.

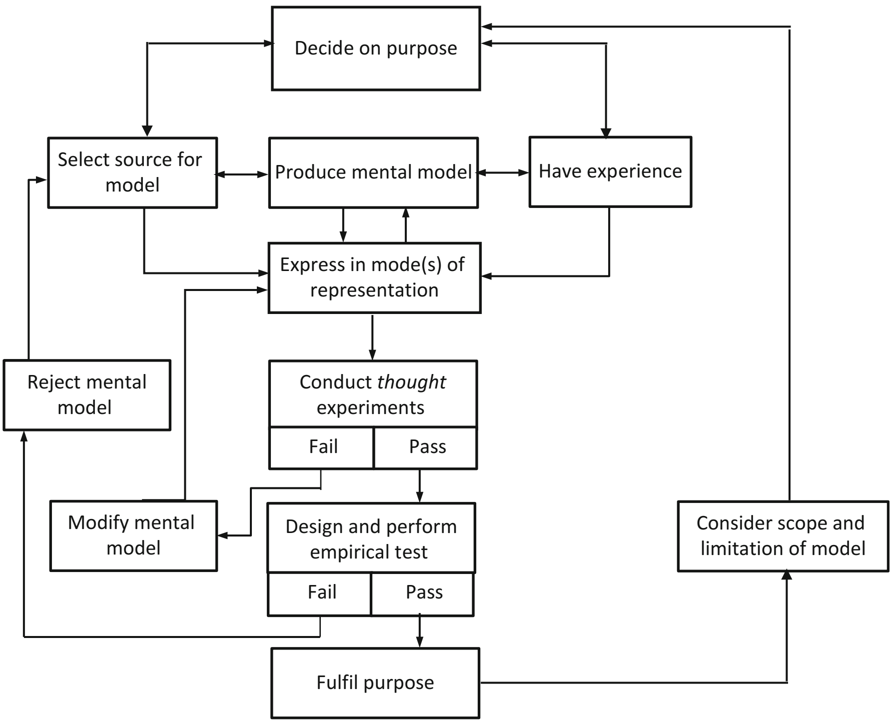

Fig. 4.2 Modeling framework in chemistry inspired by Gilbert (2002)

This modeling process is initiated by a purposeful decision that intertwines with the observation of the phenomenon recognizing its qualitative or/and quantitative properties. The next steps are the formulation of a mental model and expressing it using qualitative or quantitative representations: material, visual, verbal, mathematical (Buckley and Boulter 2000). After the model is formulated, it undergoes an exploration process by applying *thought experimentation* conducted in the *investigator mind*. It is a common practice that scientists often mentally rehearse the design and holding of empirical experimentation (Reiner and Gilbert 2000). If the model fails to produce predictions that are confirmed in the thought-experimental testing phase, an attempt is made to modify it and to re-enter the cycle.

Chemistry modeling cycle contains parallel phases found in the biology modeling which are the purpose of the investigation, data collection along with analysis, and the stage of model confirmation. As compared to biology, chemistry modeling cycle places a higher emphasis on using representations to model experiment's findings.

According to Gilbert (2004), it is still debatable what context should constitute chemistry, be taught, and how it should be taught.

4.4 Modeling in Physics Education

Physics is "the study of those aspects of nature which can be understood regarding elementary particles and laws" (McGraw-Hill Dictionary 2003, p. 1591). More specifically, physics is defined as a branch of science that studies the matter and its motion through space and time, along with related concepts such as (a) energy in the forms of light, electricity, radiation, gravity and (b) forces in the forms of gravitational, electrostatic, and electromagnetic. Physics deals with matter on scales ranging from sub-atomic particles (e.g., particles that make up the atom) to stars and entire galaxies. As an experimental science, physics utilizes the scientific methods to formulate and test hypotheses that are based on observation of the natural world. Observable patterns in physics are foundations to formulate mathematical models (Hestenes 1995). The purpose of investigations in physics is similar to these in biology and chemistry that is to make scientific laws. Physicists and especially theoretical physicists make extensive use of mathematics to communicate their findings. Theoretical physicists, as opposed to experimental physicists, construct theories expressed as mathematical models based on a deductive inquiry, starting for example, with assumptions about the motion of stars. They subsequently analyze the mathematical consequences of their assumptions to fit the model to the motion of electrons in atoms. Theoretical physicists use mathematics as a reasoning backup, and their success depends on how well the results agree with observations of nature. In case of dissonance, the mathematical representations, not the facts observed, are reexamined. A model in physics education is perceived as a surrogate object or a conceptual representation of a real purpose. Hestenes (1995) proposed a general modeling cycle in physics that is illustrated in Fig. 4.3.

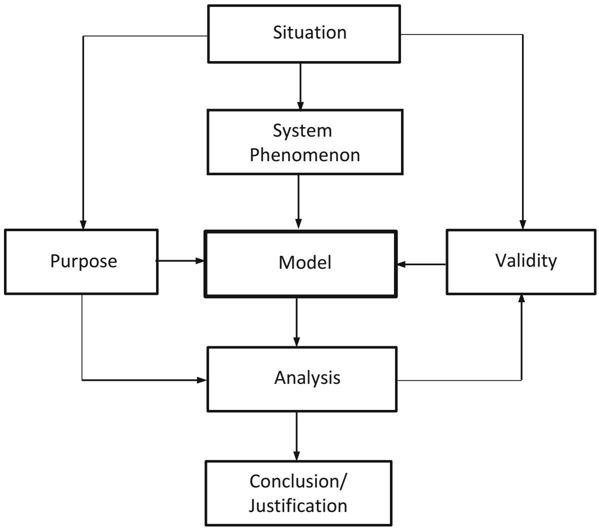

Fig. 4.3 Modeling in physics inspired by Hestenes (1995)

Real situation or experiment appears to be critical elements of modeling in physics. This cycle highlights an embedded system phenomenon that is reflected by formulated model. The stages of purpose and validity are paralleled in this modeling process, and both support the model formulation. Justification of derived model and conclusions complete the modeling process. Due to its mathematical forms, modeling in physics, unlike in biology or chemistry does not contain *dry* or mental models. According to Brewe (2008), a mathematical model has four components: (1) a set of names for the object and agents that interact with it; (2) a set of descriptive variables representing properties of the object; (3) algebraic equations of the model describing its structure and time; (4) an interpretation relating the descriptive variables to properties of some object which the model is to describe. Physical models being of mathematical forms contain all of these components.

Despite extensive use of mathematics in physics research, a supportive use of mathematics in physics education is still fragmentary and consensus on how the increase of the role of mathematics in physics is not reached (Uhden et al. 2012). Physics students fail to see more profound physical relationships in formulas and often use mathematics only within its pure algorithmic sense (Domert et al. 2012). This finding is often interpreted by physics education community as students' lack of problem-solving skills that result in a shallow understanding of the underlying mathematical and physics concepts.

4.5 Modeling in Mathematics Education

Modeling was introduced into mathematics education to bridge the gap between reasoning in mathematics and reasoning about a situation in the real world. Due to a diverse range of applications, the literature does not provide a homogeneous

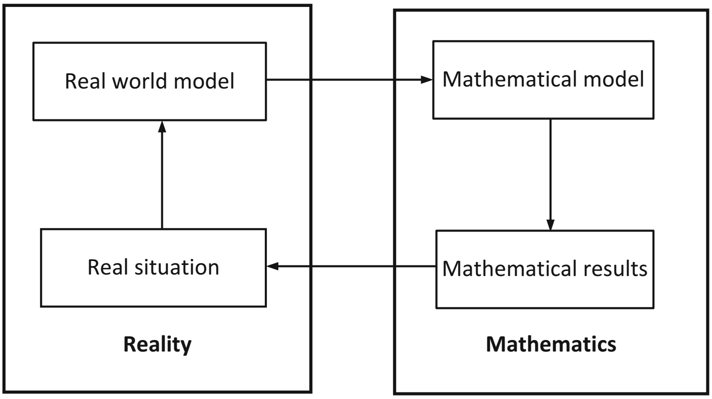

Fig. 4.4 Mathematical modeling cycle (inspired by Blum (1996))

definition of what mathematical modeling is. Traditionally, modeling is perceived as a tool supporting problem-solving or a strategy in engineering that provides opportunities for transitioning between problems and developing artifacts. Lesh and Harel (2003) defined it as finding patterns, quantifying and generalizing the patterns. Such view of modeling in mathematics also corresponds with Blum et al. (2007) perception who describe it as "learning mathematics to develop competency in applying mathematics and building mathematical models for areas and purposes that are extra-mathematical" (p. 5). More broadly, modeling is described by Blum and Ferri (2009) who situate it as a tool to (a) help students to better understand the world; (b) support mathematics learning, motivation, concept formation, and comprehension; (c) contribute to developing various mathematical competencies and appropriate attitudes; and (d) contribute to an adequate picture of mathematics.

From the epistemology points of view, Kaiser-Messmer (1996) identified three main perspectives of mathematical modeling: (1) pragmatic that focuses the learners on solving practical problems, (2) scientific–humanistic which focuses the learner on creating relations between mathematics and reality, and (3) integrative that blends pragmatic and scientific perspectives. Historically, the pragmatic view was advocated by Pollak (1978), and the scientific–humanistic by Freudenthal (1973). One of the first schematic diagrams of mathematical modeling was developed by Blum (1996), see Fig. 4.4.

The modeling cycle consists of two parallel chambers called *reality* and *mathematics,* which comprise further of building blocks defined as a *real situation*, *real-world model, mathematical model,* and *results*. By weighting reality and mathematics equally, the cycle offers a structure that is often deployed in school mathematics' problem-solving. The initial stage of the process is labeled as a *real situation*. By moving through the modeling cycle, the modelers are to return to the *actual situation* which is a solution to the given problem. The cycle is often perceived as a guide to problem-solving, and it seeks a unique solution to given real situation. It is assumed

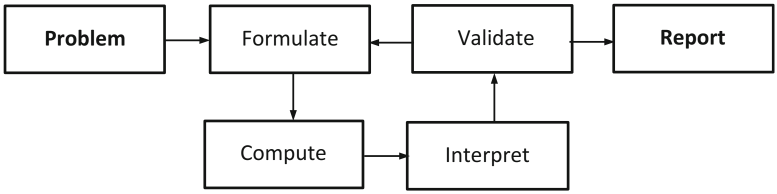

Fig. 4.5 Mathematical modeling cycle inspired by CCSRT (2013)

that the reality or nature of the problem is well understood by the problem-solver before an attempt to generate a model is made. The problem-solver is to define what the essential elements of the reality are and how to apply mathematics to solve the problem. Applied mathematical algorithms are to yield numerical values, which are further interpreted as problem solutions.

More recently, the common core standards writing team (CCSWT 2013) took a more pragmatic approach to develop a modeling cycle (see Fig. 4.5).

The modeling process is initiated by providing students with a problem who identify relations between variables and formulate a model. The next phase is to use the model to compute and the quantity of interest to validate it and interpret the results. Once the result is validated, it is reported as a concluding phase of the modeling cycle. Working through the stages, the students develop and use their critical, observational, and translational skills. There are more mathematical modeling cycles found in the literature (see Sokolowski 2015). All modeling schemes share specific standard features; they begin from a real situation/problem and conclude with a deductively attained unique solution.

While the idea of introducing mathematical modeling to school practice enhanced problem-solving, research indicates some stages that need improvement in order to have students benefit more from the modeling processes. Cobb et al. (2010) contended that despite efforts the process of problem-solving is disconnected from mathematical modeling. Klymchuk et al. (2008) and Lesh and Zawojewski (2007) claimed that exercising problem-solving in the current math curriculum does not generate the skills that it intends; students use superficial keyword methods rather than analyze embedded mathematical structures in the attempt to solve the problems. Redish (2005) argued that mathematicians and physicists have different goals in using mathematics in a sense that physicists are looking for meaning and interpretation of the equations, whereas mathematicians would most often be interested in solving the equations. This lack of connection might partially explain students' difficulties in developing mathematical reasoning skills. Thus, a need to search for such opportunities emerged, and it seems that STEM environment can serve as a field to merge these views.

4.6 Modeling in Engineering

Engineering is defined as "the science by which the properties of matter and the sources of power in nature are made useful to humans in structures, machines, and products" (McGraw-Hill Dictionary 2003, p. 723). Understanding the principles of how mathematics supports science discovery seems to be an essence of modeling in engineering. By observing how the world works, and by using technical vocabulary along with various kinds of mathematical models, engineers are interested in creating artifacts that have not yet been invented. Prior designing the devices, engineers need to be able to use scientific reasoning and model mathematically the processes. Designed and built devices will support these processes. The scientific method and engineering design have much in common, and the ability to apply knowledge of mathematics and science to design and conduct experiments and analyze and interpret data appear on the top of the requirements for students to succeed in engineering classes (Felder and Brent 2003). Are there differences between modeling in science and modeling in engineering? Yes, while science models are often applied to predict what will happen in a future situation, in engineering design the predictions are used to assure that the design will serve its purpose safely and efficiently. For example, to serve its purpose a bridge must stand or the airplane must fly.

Modeling cycles in engineering can take diverse paths depending on the structure of the final product. The diagram (see Fig. 4.6) illustrates a general route of modeling in engineering proposed by Cobelli and Carson (2001).

One of the most critical stages of the process is the second stage where the modeler needs to identify the governing physical principle of the system. The

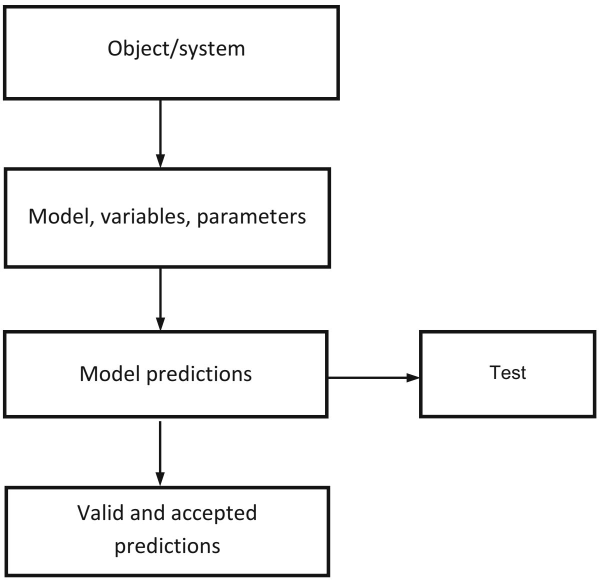

Fig. 4.6 Modeling in engineering inspired by Cobelli and Carson (2001)

product of this phase, the principle will be further applied to quantify the model outcomes. Once the model is formulated and verified, it moves to a phase of improving it. This phase can include additional variables or assumptions/restrictions that can be lifted or retained. A new modified model will undergo an interactive loop called model–validate–verify–improve–predict until it reaches a form that satisfies its purpose in the most optimal ways (Cobelli and Carson 2001). Hence, unlike in scientific modeling, engineers are looking not only for modeling the processes but also for designing the devices considering the most optimal outcomes. The motivation and general approaches to scientific modeling and modeling in engineering are not the same though, yet they share common features with science being the backbone of engineering. While the models in engineering can take various forms, for example, patterns of structure, circuits, chemical processes, and mechanical parts, the most prevailing ones are mathematical models (Felder and Brent 2003).

4.7 Technology in STEM Modeling

Technology is defined as a “systematic knowledge and its application to industrial processes closely related to engineering and science” (McGraw-Hill Dictionary 2003, p. 2109). Technology in mathematics is referred as the use of computer software to expedite computations and allow for diverse representations of mathematical objects. From the time of development of the mainframe computer in 1942, mathematicians and mathematics educators have been intrigued by the vast computational possibilities offered by technology. Technology can also mean the knowledge of techniques applied to accomplish a particular goal. Herschbach (2009) suggested that there are two common perspectives of technology: an engineering and humanity. The engineering perspective suggests that technology is equivalent to making and using of material objects (artifacts) and the humanities view of technology focuses instead on the human purpose of technology as a response to a specific human endeavor (Feenberg 2006). Among the earliest applications of the new technology to mathematical learning in schools was computer-assisted instruction (CAI)—the design of individualized student-paced modules that were to promote a more active form of student learning (Kelly 2003). The microcomputer and the graphing calculator developed in the late 1970s also supported the growth of functional approaches in algebra and interest in multiple representations of mathematical objects (Heid and Blume 2008). Graphing calculators commonly used in mathematics and science classes allow for performing a broad diversity of mathematical algorithms ranging from finding accumulation and rate of change to formulating symbolical forms of derivatives and antiderivatives. While technology is often taken to be a reference to digital technologies, its meaning according to Feenberg (2006) is instead associated with the epistemology of inducing engineering modeling in practice or means of achieving the goal of engineering modeling. While modeling is the primary method of learning sciences and designing in engineering, the cycles and the aims of research illustrate that the paths taken to employ technology are not identical, yet in all, technology is a tool to support the investigations.

In the proposed STEM modeling projects in this book, engineering perspective of technology will be explored and will be used as a means to enhance quantification of data and support the verification of mathematical representations and structures that the learners will develop.

4.8 Synthesis of Modeling Techniques

While biology uses extensively simulated environments before embracing the process with real experiments, chemistry emphasizes thought experiments as a prelude to actual experimentation. Theoretical physics, on the other hand, highlight the use of the tools of mathematics to prove or disprove their hypotheses. Engineering modeling extends the pathway; it not only includes the stages of scientific modeling but it moves the process to a next level of designing artifacts that would reflect on the rules of the phenomena. Mathematics in these modeling processes takes a position of a *guardian* that guarantees that all phases of the modeling processes are quantified correctly.

In the three fundamental disciplines, science, mathematics, and engineering, technology is to support quantification processes and thus enhance mathematical reasoning. Technology supplies the tools to perform the quantifications more efficiently or to produce alternative visualizing means of the experiment outputs or the anticipated artifacts. All the modeling processes share standard features that are illustrated in Fig. 4.7.

The survey of the modeling techniques revealed that one of the most critical elements of engineering design is the designers' ability to apply mathematical reasoning and scientific inquiry to model the phenomenon and then to design an artifact. Mathematical modeling is of particular importance because its quantification methods produce viable means for enacting and predicting how newly designed artifacts will behave in new circumstances. Technology supports the quantification process, however, selecting a mathematical structure *of the best fit* depends on to designers judgment and thus his/her mathematical skills. STEM provides excellent opportunities for developing these skills in mathematics school practice. Intertwining the primary purpose of developing mathematical reasoning with scientific inquiry and respecting the fact that disciplines mark distinctive methods of knowing appears as a highlight of the design process. While a deductively organized problem-solving in mathematics stands at odds with inductive science, it seems exploratory character of STEM projects provides opportunities to link these epistemologies into one cognitive task.

Research shows that students' perception of mathematical modeling in college is greatly affected by their prior educational experiences with modeling. Staats and Robertson (2014) claimed that students' difficulties with modeling in mathematics are due to being new to such activities when they enter science and engineering

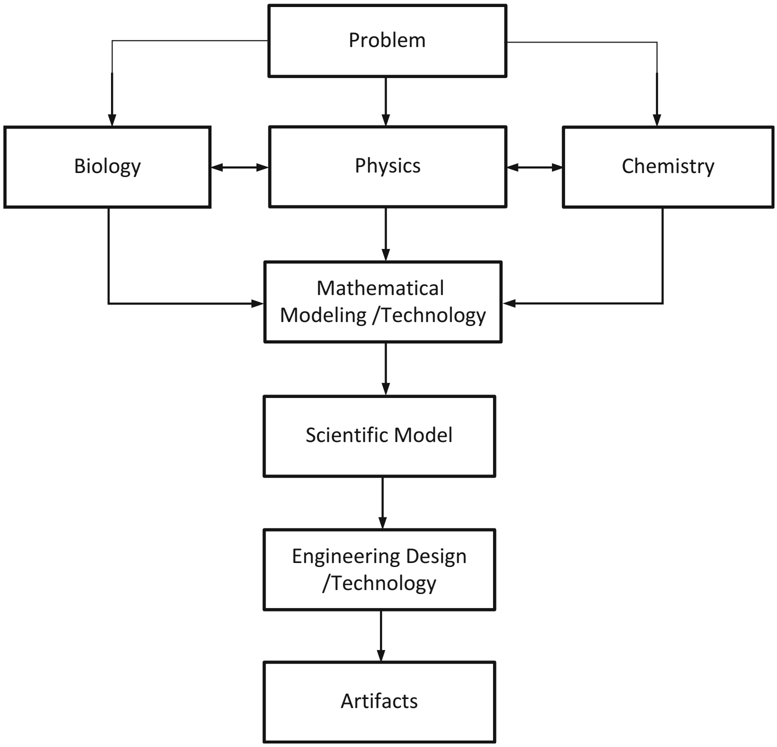

Fig. 4.7 Phases of STEM projects

college programs. If problem-solving is followed by developing mathematical reasoning and the techniques of modeling, the prospect of producing successful modelers is much more promising. Making STEM modeling a priority in a high school curriculum seems to be one of the actions supporting this premise.

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Chapter 5
Survey of the Field of Empirical Research on Scientific Methods in STEM

Abstract Research shows that being versed in scientific modeling is a precursor to succeed in engineering modeling and might be a factor attracting students to engineering. This finding suggests that STEM activities that develop the skills of modeling should not only focus on students' mathematical and scientific reasoning skills but also provide an environment where students would feel comfortable and encouraged to continue these enterprises in their college and professional careers. One of the main obstacles in adopting inquiry-based learning projects in mathematics is the gap between problem-solving in mathematics and inquiry in science. It appeared worthy to search the literature on STEM education to determine what the research findings in this domain are. The synthesis of the survey findings will support multidisciplinary STEM framework developed in Chap. 6.

5.1 Formulating Criteria for Literature Review

There were several limitations in determining the review search criteria. One of them was a broad scope of the objectives and terminology used to describe STEM research. To overcome this limitation, guidance that assured transparency of the terms and selection of scientific methodologies was established. Although I realized that due to that generalization, the synthesis might not be entirely comprehensive, I expected that its nature would locate enough studies that would reflect on the search aim. The primary goal of the survey was to extract findings of what phases of scientific methods are included in STEM activities and how they affect students' learning. More specifically, this survey centered on gathering STEM activities that included the inquiry type, hypothesis formulation, data gathering, model development, and the validation process.

With this aim established, the following search questions were formulated: (1) *Are the elements of scientific methods such as hypothesis stating, the type of inquiry, the stage of model eliciting and its validation existing in STEM activities*? (2) *How do students merge the mathematical and scientific reasoning and what are the main challenges and research recommendations to eliminate these challenges?* The survey encompassed studies conducted at high school and college levels.

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The source of data collection was a systematic review of literature written in English language and published in peer-reviewed education journals between January 1, 2000, and March 30, 2017. The survey considered each study as individually informative (Hsieh and Shannon 2005) and was led by coding and identifying patterns.

The following emerged as critical terms: (a) *Scientific inquiry*—a method or procedure that consists of experiment, hypothesis, and systematic observation along with formulation, testing, and modification of the model; (b) *STEM education*—activities and projects that integrate science, technology, engineering, and mathematics; and (c) *Mathematical modeling*—process of developing a mathematical model. One of the other vital terms was *explorations* defined as an act of searching to discover information. Learning via multifaceted explorations plays a vital role in developing students' skills in science and mathematics classes (e.g., Pollak 1978). Explorations are also the essence of modeling in engineering (see Sect. 4.6). The primary research was conducted using ERIC (Ebsco), Educational Full Text (Wilson), ProQuest, and Science Direct. The following queries were used: (a) "STEM activities" OR "scientific inquiry" AND "college" AND "mathematics"; (b) "Mathematical modeling" AND "high school" AND "scientific methods"; (c) "Multidisciplinary projects" AND "scientific inquiry." While this search led to 213 studies, given the previous argument, I have considered, that even using these terms collectively might not capture all literature relating to this synthesis. Many of these studies pertained to a general purpose of STEM education, curriculum and testing constraints, and professional development for teachers. The scrutiny that aimed at extracting only exploratory activities revealed that 32 studies that represented 18 countries satisfied the study criteria. Considering a large time interval, this rather small range of studies, showed that investigating the effects of the elements of scientific methods on students' mathematical reasoning is not widely practiced.

5.2 Synthesis of General Findings

A systematic review can take different paths; it can be qualitative, quantitative, or it can take a form of being a combination of these two aims. Because the primary goal of this undertaking was to learn *how* the elements of scientific inquiry have been used in STEM activities, this review focused on synthesizing qualitative research. A post hoc analysis has shown that in many of these studies ($N = 63\%$, 20) students initiated STEM activities from designing artifacts, e.g., bridges, rockets, without a limited prior exploration of the underlying scientific principles prior (see, e.g., Kertil and Gurel 2016). The goal of such projects, thus building artifacts, reduced the algebraic apparatus to simple algorithms, and the contribution of science concepts was respectively reduced to applying students' *naïve* thinking. In the remaining part of the pool (38%, $N = 12$), only certain phases of scientific methods were applied. It is hypothesized that focusing students' attention on constructing artifacts that require using algebraic algorithms might be one of the reasons for which students do not perceive mathematics as a subject helping them to reason, but as a subject providing them only the supportive tools for quantifying. More specific conclusions on how the processes of STEM activities are organized, as seen through the prism of scientific methods were clustered in four subchapters that follow.

5.2.1 Findings of How Students Formulate Hypothesis

Although research shows that stating hypotheses benefit the learning, students were expected to formulate hypotheses only in four ($N = 4$, 13%) of the studies (e.g., Faraco and Gabriele 2007). Authors of these studies pointed out weak students' skills in formulating and proving or disproving the hypotheses. In ($N = 2$, 6%) studies, participants were supposed to formulate hypotheses quantitatively and test mathematical concepts (Lim et al. 2009). The scientific nature of these activities was not explicitly brought up to students' attention.

Hypothesis reflects closely on problem statement or the purpose of the investigation. Once formulated, a hypothesis focuses the investigator's attention on a narrower area of inquiry and helps the investigator extract information necessary to prove or disprove it. The hypothesis can be perceived as the researcher's proposed theory explaining why something happens based on his/her prior knowledge (Felder and Brent 2003). A well-established precept in education is that a strong motivation to learn is generated by a strong desire to know (Albanese and Mitchell 1993). Thus, formulating hypothesis can also be perceived as a motivational factor to drive students' motivation to complete a given task. Hypotheses play central roles in the process of modeling, and students' difficulties in its formulation may also reflect on their weak STEM reasoning skills. The hypothesis in STEM modeling activities are suggested to be of verbal forms and should enable testing mathematical and scientific concepts. This duality will require establishing a balance between mathematics and science contexts during the lab conducts. It is doubtful that reducing problem statements and consequently the hypothesis to formulating only mathematical representations will nurture the connections between a symbolical mathematical language and scientific contexts and help students develop STEM reasoning. Constant mediations between scientific contexts and algebraic representations will lead to establishing an equilibrium to construct new knowledge and support the development of students' STEM reasoning.

A need for more elaboration is also needed to differentiate between hypothesis and prediction. As hypothesis proposes an explanation for some puzzling observation, a prediction is defined as an expected, often as a numerical outcome of a test of some elements of the hypothesis (Lawson et al. 2008). Prediction also refers to model's ability to foresee, with a certain degree of accuracy, what will happen in the future conduct (Bechtel and Abrahamsen 2005).

It is critical that in STEM activities, the hypothesis targets not only embedded mathematical structures but also scientific principles rooted in the activity.

For example, if an activity is about optimizing the profit of producing and selling goods, the mathematical structures used to model this context will most likely be polynomial functions along with determining their intersecting points, while the scientific principle will be represented by the law of supply and demand. If the activity is to model projectile motion, then mathematical structures would include polynomial parametric equations of the first and the second degree, and the scientific principle would be represented by the properties of object's motion in the gravitational field. Furthermore, if students are to derive Newton's second law of motion during STEM mathematics activities, then the hypotheses will attempt to find out what type of algebraic function can be used to describe mathematical dependence between an objects' acceleration and the net force acting on them. In physics classes, hypothesis targeting this idea is typically reduced to investigating how an object's acceleration depends on the net force acting on it, and a general statement of proportionality is sought. It is proposed that hypotheses in STEM projects are inclusive to disciplines involved in the given STEM project.

I propose using multidisciplinary STEM activities as opportunities for the students to revise and deepen their mathematical and scientific reasoning skills through reflecting on carefully formulated questions embedded in the instructional support. Highlighting the role of hypothesis, its structure and formulation in STEM projects are one of the areas that the proposed theoretical framework will address in more details.

5.2.2 *Inquiry Types Used During STEM Projects*

There are three types of inquiry: inductive, deductive, and abdicative. While the inductive reasoning is typically used in sciences, deductive is used in mathematics and engineering (see Chap. 4). An analysis of the located studies revealed that *inquiry types* applied during the activities were not discussed and brought up during the activities' designs. The literature also did not provide a coherent view of what inquiry type—inductive or deductive—is suggested for STEM activities. In several of the studies (Soon et al. 2011), students applied algebraic equations to find specific solutions or worked on problem-solving using, e.g., interactive media (Liang et al. 2010; English and Gainsburg 2015) which implies a deductive approach. This method also prevailed in undergraduate STEM programs (Williams et al. 2015).

Observation, analysis, and model formulation appeared to be crucial elements of scientific methods with the model to be regarded as an ultimate product of these phases. These phases though are not explicitly highlighted in STEM activities. Because the general method of reasoning affects activity design and goals, a need for discussing it emerged. Mathematics is traditionally considered a deductive discipline as opposed to science that requires reason inductively. Inductive reasoning has been proven to play a significant role in concept learning and the development of mathematical expertise (Haverty et al. 2000). The motivational and learning benefits of teaching inductively are reported throughout the literature about cognitive and educational psychology as one of the most efficient pedagogy (Felder et al. 2000), and this reasoning will guide the proposed activities.

5.2.3 *Concerns About Extracting Variables and Model Formulation*

One of the major concerns voiced in the accumulated research was students' inability to transfer given problem or process of an experiment into mathematical representation (Soon et al. 2011). This phase is essential in modeling, and a deeper analysis of this deficiency is necessary to suggest improvements. While searching for more prompts, more detailed questions about the origin of the difficulties emerged such as is this deficiency due to a weak student understanding of mathematical structures (e.g., the properties of periodic functions, the differences between rate of change and a percent rate change) or is the deficiency due to difficulties in identifying conceptual patterns in given problem/experiment and mapping the patterns on corresponding mathematical structures? If students cannot identify algebraic structures that would reflect a system's behavior, then the reason for the deficiency might be a lack of mathematical knowledge or skills in applying the knowledge in real situations. If the difficulty lies in recognizing the properties of system behavior, then this difficulty can be attributed to a lack of scientific inquiry skills or lack of the content knowledge embedded in the activity. Several research studies suggested that to succeed in STEM modeling activities students need to possess necessary background about the concepts/laws that they are to integrate. This suggestion requires that instructor needs to assure that students had learned contents of the disciplines prior having them integrate the knowledge. Expecting that the students will learn and simultaneously integrate the knowledge might be misleading. Such arrangement might lead to seeing students confused and discouraged. Jonassen (2011) concluded that going into a problem or experiment with a vague goal of figuring it out is unlikely to lead to a meaningful solution. The skill of developing scientific inquiry is not naturally possessed by individuals, but it can be developed through carefully designed instruction. Developing STEM reasoning using modeling is challenging for students, and instructional support is necessary. Even if students had learned the contexts prior a modeling activity, the interface of integrating of the two worlds—real situations and their corresponding mathematical representations—requires specific prompts that can be formulated by the instructor. Some of such prompts proposed in this book are (a) establishing duality of the hypothesis, (b) intertwine mathematics and science ideas throughout the lab, and (c) designing the verification process of the elicited model that would validate not only algebraic structures but also the science outcomes of the experiment.

Students' difficulties in deciding about cause and effect, formulating variables, and classifying the variables to develop a mathematical representation are being discussed in research (e.g., Diefes-Dux et al. 2012), however, seems to be done in this regard. While the categorization of variables as dependent and independent is signified when labeling the values in the Cartesian plane or in using the function notation, deciding which quantity is dependent and independent causes doubts when realistic contexts are presented, and the variables are not explicitly picturized. These doubts can be reduced, and the skills of classifying variables improved, if a discussion of the purpose of the lab and its general scientific underpinnings are discussed with the students prior the lab conduct. Is there parallelism of *cause* commonly understood in mathematics as the independent variable and *effect* as the dependent always respected in sciences? Many scientific discoveries prove that this association is not always reinforced; in fact, there are several laws in physics, for example, Ohm's or Hooke's laws where the *cause* and *effect* do not correspond with the meanings of independent and dependent variables as defined in math classes. These modifications in science are done purposely. Explicitly informing students about these modifications would broaden the perspective and encourage flexibility.

Another area worthy of discussing is a classification of quantities as *given* and *required,* often used in science, and the relation of this classification to mathematics problem-solving in STEM contexts. When STEM project is about finding a unique solution to a given problem, classifying variables using that manner might be helpful; however, in exploratory STEM projects, it might not be sufficient. In fact, the research showed that classifying quantities as *given* and *required* constitutes a preliminary step to succeed in STEM multidisciplinary modeling projects (Lim et al. 2009). This finding suggests that dynamic STEM systems require more sophisticated instructional guidance in identifying and classifying variables. Based on the research findings, students do not transfer their scientific inquiry skills to mathematics classes automatically. Working on STEM activities in mathematics classes is methodologically different from the once students encounter in their science classes. Thus, the commonly used scientific classification of quantities as *given quantities* and *required quantities* needs to take another more sophisticated meaning in STEM projects—it can be perceived as identifying essential function parameters (e.g., slope, coordinates of the vertex, function period, initial value, rate of change, and so forth). Identifying these attributes in the given experiment will help in using a correct function, and consequently a correct mathematical representation. Taking data to construct algebraic functions to model system behavior does not constitute the final stage of collecting evidence displayed by the system; the modeler needs to extract specific attributes of the system behavior and map the attributes into known algebraic structures. In the STEM studies that focused students' attention on building artifacts, this step was omitted. It is seen that to help students reason like scientists, extracting these differences from the epistemology of mathematics and science points of view and giving students a chance to discuss these meanings when applied in STEM environments will be helpful. Students must be given opportunities to realize the relevance of function features not only in static—content-free—problems typically found in mathematics textbooks but also in dynamic scientific contexts. Mentzer et al. (2014) claimed that "Teachers should seek opportunities to demonstrate the value of mathematical modeling and encourage students to think about relationships and functions as ways of understanding the natural phenomena" (p. 313). Thus, to have students gain meaningful learning experiences from multidisciplinary STEM activities guidance explicating on how to use learned abstract concepts in real contexts is necessary.

5.2.4 Concerns About the Validation Process

Validation phase whose purpose is to determine how well the derived algebraic representation resembles the system behavior was undertaken in eight of the studies ($N = 8$, 25%). The low frequency perhaps reflects on the deductive inquiries that had been applied in most of the gathered research studies that typically seek unique solutions. Several lobbyists (Klymchuk et al. 2008) noted that students failed to validate formulated mathematical structures or had difficulties with the derived model's contextual interpretation. Crouch and Haines (2004) claimed that the transitioning from formulated model back to the real-world problem is the most challenging phase of the modeling process. These findings suggest that more prompts should be generated in the instructional support to guide modelers about the validation of the elicited algebraic forms. The mathematical representation should be perceived as an explanation that refers to the ability of the model to account, in sufficient detail, for the underlying causal mechanisms that produce some observed outcome or phenomenon (Bechtel and Abrahamsen 2005). The purpose of the formulated models is not only to verify adherence of their forms to the conventional structures but also to assure that their forms can be used to learn more about given phenomena in new situations, for example, while solving textbook problems of working on the STEM projects. To better align mathematical and scientific methods, the interpretations should mediate throughout the processes and should produce valid conclusions. Marginson et al. (2013) and Honey et al. (2014) concluded that mathematics and science concepts and representations must be explicitly flagged and engaged with, and linked across activities so that cohesive ideas and relations through the design process are not left to chance. It is believed that designing multidisciplinary STEM projects in such structure will help develop an epistemology that will parallel not only with inductively organized inquiries in sciences but also support and develop mathematical reasoning.

5.2.5 Interface of Problem-Solving and Modeling in STEM

Developing problem-solving expertise in students has become a central concern of science and mathematics education researchers and practitioners (Diefes-Dux et al. 2012). Problem-solving can be embedded in any STEM projects, and in fact, STEM projects often are about solving a problem. However, due to multiple voices from the research STEM community about students' difficulties in problem-solving, other routes of addressing this issue are sought. How can STEM modeling activities proposed in this book contribute to these efforts? While the relation between exploratory STEM activities and problem-solving has been discussed by mathematics community, the effect of the order of sequencing these two enterprises on students' learning has not been explicitly examined. In fact, the two areas, problem-solving and explorations, appear disjointed in the current research and practice.

Although multidisciplinary STEM projects are to provide general inquiry techniques not only in mathematics, the fundamental question to answer is if STEM modeling in mathematics should follow problem-solving or if problem-solving should follow modeling? Alternatively, should both be considered separate practices, or should one follow the other? Some researchers (e.g., Mousoulides et al. 2008) defined problem-solving as repetition of procedures, and modeling activities as context-rich problems that do not assume that students have already learned the procedure for solving the problem. While some educators consider modeling as a distinct activity from problem-solving, others have proposed otherwise and the view depends on the purpose that a specific word problem or a modeling activity serves in the given learning context.

Modeling in this book is perceived as an exploratory type of activities giving students opportunities to merge scientific inquiry skills with mathematical reasoning, thus providing the novice modelers with a background and directions on how to approach problems using scientific methods. Modeling is to equip students with systematic methods to solve the problem and use these methods or some of their phases to solve other similar tasks. With this purpose assumed, exploring properties of phenomena will not constitute problem-solving per se but rather it will constitute a prelude to a subsequent problem-solving rooted in similar contexts. This pathway corresponds to how science and engineering modeling processes are related (see Chap. 4), and it is also supported by research. A study by Lingefjärd and Holmquist (2005) revealed that after working on modeling activities that are usually supplied by real embodiments, students handled word problems better than those who were taught by conventional methods. Yu (2011) concluded that "developing the modeling ability promotes students' problem-solving ability" (p. 152). Dean and Kuhn (2007) highlighted the importance of prolonged opportunities for implementing the control-of-variables strategy during the solution of extended problems because such opportunities convert the meanings of formulas into algebraic functions that can be used to uncover multiple solutions. Research in engineering education explicitly stated that students need to acquire modeling skills prior engaging in engineering designs because interpretation and understanding of the context of a problem are critical for constructing meaningful and adequate mathematical representations of covarying quantities (Moore and Carlson 2012). Modeling coupled with explorations is to develop contextual background and STEM reasoning skills to transfer to a narrower area of problem-solving that seeks to solve specific cases.

While there is robust research supporting the thesis that carefully designed modeling environments can serve to foster and solidify students' problem-solving skills, research on how to make the transition from STEM exploratory modeling projects to problem-solving that benefit the learning is limited. In the proposed multidisciplinary STEM modeling cycle (see Fig. 6.1), the phase of model verification during which the learner uses the derived mathematical model to answer additional questions is to link modeling to the typical word problems found in textbooks. Viewed through this prism, problem-solving perceived in this book is an extension of modeling activities and it will constitute its integrated part that is nested in the verification phase of the scientific inquiry. The epistemology of such sequencing is illustrated in Fig. 5.1.

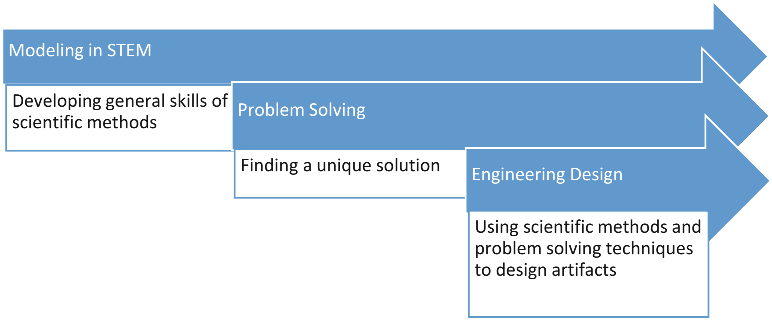

Fig. 5.1 Modeling and problem—solving sequencing within engineering design

Niss (2010) contended that knowing mathematical theories does not guarantee that this knowledge is transferred automatically to abilities to solve real-life problems. This finding suggests that students need a transitioning phase from mathematical theories to context-rich problem-solving and it seems that STEM modeling can fill in this gap.

While modeling has proved to be helpful with problem-solving, the question that needs more research to be answered is what phases of the modeling process are particularly beneficial for developing students' problem-solving skills. Problem-solving in science can be enhanced by inducing an engineering design approach because it creates an opportunity to apply science knowledge and inquiry as well as provides an authentic context for learning mathematical reasoning. Problem-solving in mathematics, using STEM environments, exemplifies the algebraic structures making them not only as algorithms but also as means to justify the scientific reasonableness of the derived models. While this might be a premature statement, it seems that the degree to which modeling activities can support problem-solving skills is rooted in their capacities to extend students' analytic skills and abilities to apply these skills and mediate them with new contexts.

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Chapter 6
Formulating Conceptual Framework for Multidisciplinary STEM Modeling

Abstract This chapter builds on the accumulated research findings of previous chapters and proposes a theoretical framework to design multidisciplinary STEM activities whose main learning goal is to develop students' mathematical and scientific reasoning skills. Research showed that mathematics in the STEM is underrepresented; therefore, enhancing mathematical reasoning was given a priority. Also, the theoretical framework aims to have students mathematize phenomena and use the structures to support problem-solving techniques. This chapter explains the structure of the framework (Fig. 6.1) along with how the links between mathematical and scientific concepts will mediate. While the framework is proposed for mathematics classes, it can also be used to design activities in other STEM disciplines. The framework was used to develop activities that are described in Chaps. 7, 8, 9, and 10.

6.1 Framework for Multidisciplinary STEM Learning

The modeling scheme (Fig. 6.1) emerged from the synthesis of the STEM research. It reflects on recommendations that called for mathematics and science ideas to be explicitly highlighted, engaged with, and linked in STEM activities to assure a broader and more cohesive blend between these two disciplines. The suggested cycle is an enriched version of one that was previously developed to stimulate mathematical modeling (Sokolowski 2015). While the earlier framework focused on supporting mathematical modeling, the proposed herein is more comprehensive. It opens a gateway for further explorations to exercise problem-solving and to develop independent projects or design artifacts. Mathematics in this process is to (a) provide a means to generate concise symbolic representations and support understanding of mathematical ideas; (b) serve as a source of tool to quantify scientific concepts, and (c) help extend the analysis of the phenomena beyond the classroom constraints. Science, on the other hand, is to provide quantifiable contexts. I acknowledge that organizing any STEM activities using the proposed framework might not be possible because of some areas of mathematics and science knowledge carry out a high theoretical load that might not provide sufficient measurable data to be embraced in the modeling processes.

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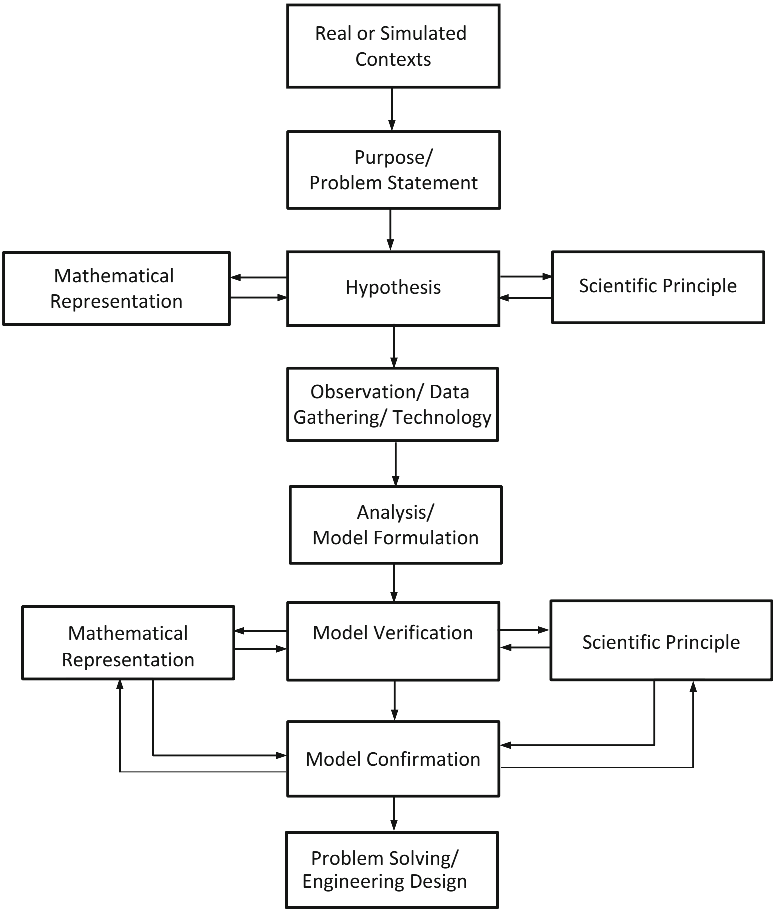

Fig. 6.1 STEM modeling process supporting scientific and mathematical reasoning

From an epistemological point of view, the cycle represents a pathway of learning that focuses on contextualizing the tools of mathematics through science embodiments. It was anticipated that the learning pathway would encourage the students to create new knowledge about the systems under investigation and have them realize that to provide a scientific answer, a single piece of information is not sufficient, and that experiment needs to be designed and data collected and analyzed. It was also hoped that students would realize that each experiment has certain limitations and that most knowledge is not absolute. Perkins (2004) claimed that understanding is a matter of being able to do a variety of thought-demanding things with a topic, like predicting, finding evidence, generalizing, applying, and representing the subject of understanding in a new way. The pedagogical prompts of the proposed enterprise intend to encompass these tasks, and foremost, they are to convince the modelers that inquiry embraced in STEM modeling is one of the ways of knowing and understanding. As the modelers become more acquainted, the design might be simplified leaving more room for students' creativity and invention.

The modeling process consists of phases of scientific methods and auxiliary tasks that explicitly target multidisciplinary nature of STEM contexts. The subchapters that follow explain how the central phases of the process were assembled and how they intertwine to engage students in the integrated math-science inquiry process.

6.1.1 Inquiry Type and Students' Reasoning Skills Development

National Research Council (NRC 2000) characterizes a complete inquiry cycle as comprising an identification of questions, the design of an investigation, examination of empirical data, and drawing and justifying inferences. These skills develop gradually in the context of rich practice. They require time and careful planning. STEM activities are to merge contexts of component disciplines in one enterprise. There are three main types of inquiries that STEM enterprises can be organized and the participants can develop: inductive, deductive, and abductive. While each possesses own right to be applied, the selection depends on the problem formed, the general framework of the analysis, and more importantly the type of inquiry skills the students are to practice. Following the modeling cycles discussed in Chap. 4, it was noticed that research and learning in sciences are dominated by an inductive inquiry, whereas mathematical modeling and engineering is predominantly supported by a deductive inquiry. The purpose of the suggested STEM activities is to immerse the students in the processes of generalizing natural phenomena using available mathematical systems. Students will observe the phenomena, gather data, and use the data to formulate algebraic representations. These tasks represent typical phases of inductive inquiry. Therefore, inductive inquiry emerged as a leading type of STEM reasoning suggested in this book. Inductive reasoning also plays a significant role in problem-solving, concept learning, and the development of mathematics expertise (Haverty et al. 2000). Inductive reasoning can be encouraged in a wide range of instructional methods such as inquiry learning, problem-based learning, project-based learning, case-based teaching, discovery learning, and just-in-time teaching (Prince and Felder 2006). STEM activities that can be embraced in discovery-type instructional methods (English and Sriraman 2010) further supported inductive reasoning. While the inductive inquiry is suggested in this book, the deductive inquiry can also be applied especially during these STEM projects that focus students' attention on a narrower part of analysis seeking a unique solution to a problem or build an artifact. Hestenes (2013) suggested that students must be engaged in scientific inquiry so that they learn how to shape and justify rational opinions on their own. STEM activities are presenting excellent opportunities for developing such justifications.

6.1.2 Criteria for Contexts Selection

It is recommended that the STEM contexts designated to intertwine mathematics and science be exploratory and must provide means for verifying derived model within an algebraic domain of the phenomenon. Because students do not spontaneously make connections between various disciplines, prompts for connecting their multidisciplinary elements are to be made explicit. For example, if students construct quadratic functions to model a path of a projected object, then they should be made aware that this is possible because objects undergo a constant acceleration that is $-9.8\ \text{m/s}^2$ while moving in the gravitational field. Additionally, if they are to verify enacted algebraic representation of the path of motion, they should be able to do so referring to the time of the observable motion.

The complexity of the contexts depends on several factors such as (a) the grade level taught, (b) activity purpose and learning objectives, (c) class time allocated for its completion, and (d) availability of equipment/multimedia. All these factors need to be considered so that participation and readiness of all students is enhanced. The most commonly used STEM contexts are these provided by real experiments where students can manipulate on the cause and effect and simultaneously observe the outcomes and gather quantitative data. Such environments expose students to some parts of reality that can be remembered and further deployed to word problems of similar situations. Unfortunately, mathematics classrooms are not typically designed for conducting real experiments. Therefore, a necessity to find an alternative means of conducting experiments in mathematics classroom emerged, and it seems that using simulated experiments can be a suitable replacement. Research (Podolefsky et al. 2010) shows that simulated exploratory type experiments can suffice well real experiments and allow for multidisciplinary inquiries. Education multimedia technologies that create interactive learning environments also support the development of content-specific knowledge and skills. Software, in the forms of simulations, graphing, and computational programs are set up to use various representations to visualize knowledge that is typically not observable. Learners should encounter diverse ways of explaining in discipline because the diversity of knowledge representations will help them understand and retain that knowledge (see Sect. 3.1–3.3).

Out of four proposed activities in this book, two will be supported by simulated experiments. I would like to encourage the use of scientific simulations as a means for context and data source because of their convenience to bring to the classroom and because such environments are aligned with contemporary STEM modeling cycles (see Chap. 4). The medium organization, and especially its design due to explicit purpose, plays a significant role in detecting the cause and effect, thus identifying and classifying the variables. Due to being conducted in mathematics classes, the scientific contexts of the experiments along with data taking must be simple and thus limited to exploring one concept so that the students can focus on the most important features without a need to refer to an extensive science background. A word of caution is needed here; one must realize that the medium or the context itself will not generate learning because concepts, principles, and ideas do not reside in physical materials or classroom activities but in what students do and experience (Noble et al. 2001). Thus, a careful problem formulation, coupled with a low, yet contextually rich cognitive load, is a prerequisite for developing an engaging and productive STEM learning environment.

6.1.3 Challenges with Problem Statement and Hypothesis Formulation

The catalyst for initiation of STEM activities in this book is the *Purpose* or *Problem Statement* followed by *Hypothesis* formulation. In many of the research studies (see Chap. 5), problem formulation was assigned as students' task. While there are different views on who the instructor or the modeler should formulate the problem statement, I suggest that the purpose and the problem statement are formulated by the instructor, and the task framing hypothesis is assigned as a students' task. Supplying students with a problem statement will help them with identifying the independent and dependent variables and formulate the hypothesis, thus setting the students on the right path of the lab completion. I also suggest that prior the activity conduct, the instructor explains the purpose/problem of the activity and demonstrates the general physical process of the phenomena (simulation) if possible. Research shows (Dean and Kuhn 2007) that helping students to identify a question has significant effects for the following phases of the inquiry because tasks encapsulate the goals and means by which students encounter the issues framed as relevant in the discipline. Thus, if the learner falsely identifies the *problem* embedded in the given context, he/she might not be able to extract the *cause* and *effect* accurately and will likely misinterpret the hypothesis that might jeopardize the entire inquiry process. There is a room for students to revise the inquiry and the analysis; however, limited by time constraints, the goal of the labs is to prepare the learning pathways in a manner that will equip all students with stimuli to learn from it.

What forms can a problem statement take? Problem statements are formulated through establishing precise control of the context. Due to being exploratory, problem statements should be open ended, and they refer to students' prior knowledge and learning experiences. For example, what type of function will model the maximum height of a bouncing tennis ball? Alternatively, what type of system of equations will model the motion of two cars that move on the same time domain? In the suggested activities, students will be given multiple problem statements targeting not only math concepts but also scientific contexts, and respectively, they will formulate multiple hypotheses.

The hypothesis is formulated based on an understanding of the problem statement, and it is to reflect on the activity purpose and design. To merge mathematical and scientific reasoning, the hypotheses are set to target not only mathematical structures but also scientific principles. For example, if the context of the activity pertains to modeling a periodic function, then the sought algebraic model is most likely a periodic trigonometric function, while the scientific principle can represent the properties of energy transportation using wave or a population of species that change periodically. The two contexts will merge during STEM activity process. The degree to which a derived model correctly depicts the phenomenon will be verified by their mathematical and scientific context coherence. Since hypothesis formulation depends on students' prior knowledge, it is imperative that the instructor check if students own adequate prior knowledge. Research shows that an engaged learner is inspired to accomplish the desired goals even in the face of difficulty (Schlechty 2001). The learner will be engaged if he/she possesses a necessary discipline-specific backgrounds that are to be merged. Since hypothesis is the best-educated guess formulated by prior experience, I suggest making explicit to students that their hypotheses might be incorrect. The lab purpose and subsequent conduct are to prove or disprove the hypothesis, thus have the learner confirm and correct the prior knowledge. Presenting the meaning of hypotheses in such manner has an essential effect on supporting students' self-confidence and motivation to engage in more such labs. The students also need to realize that STEM activities are to help them gain understanding and develop reasoning skills to handle problem-solving in broadly defined contexts. Designed for an average ability student, the activities are also to encourage students to apply their creativity so that they will consider becoming designers of similar projects in their professional lives. Learning environments that provide opportunities authoring disciplinary knowledge are apt to increase the likelihood that students will develop interests and dispositions consistent with disciplinary engagement (Cobb et al. 2003).

6.1.4 Analysis, Generalization, and Algebraic Model Eliciting

The importance of what it means to know is highlighted by the distinction between learning about the products of discipline—what is often called content knowledge—and learning how disciplinary products are formulated, modified, and sometimes ultimately abandoned if the derived model cannot be validated using data within the experiment domain. Tasks such as *Analysis*, *Generalization,* and *Model Formulation* are essential because they present opportunities to disclose to students how multidisciplinary products are generated and validated. While the modeling cycles reviewed in Chap. 4 did not explicate on how scientific and mathematical natures of the produced representations intertwine, the proposed STEM framework suggests that both types of representations engage throughout the entire lab conduct. This intertwining is ensured by including questions that not only prompt the modelers to quantify the phenomena but also questions that prompt the modelers to verify the adherence of the context under investigation to these quantifications. These questions are supposed to evoke mathematical reasoning and simultaneously blend it with scientific inquiry. The phases of *analysis* will contain such designed questions. As during the generalization and eliciting an algebraic representation the learner will focus more on mathematizing the phenomenon, the phase of *Model Verification* will direct the learner toward verifying independently scientific and mathematical natures and then claim their adherence to correctly or incorrectly depicting a STEM phenomenon under investigation. The mathematical structure will help quantify the values of the variables, and embedded scientific principle will justify the reasonableness of the answer. Both structures should produce a coherent model. If one does not support the other, revision of the model should be enacted. This phase will also highlight the role of the hypothesis and its mathematical-scientific duality. Once both natures of the model are validated, the learner will further test its applicability, for example, in solving congruent textbook problems. Explicitly emphasizing the duality of the audit process will also provide learners with opportunities to revise their knowledge if necessary. Structure-wise, the duality of verification parallels with the duality of the hypothesis stating and it warrants reusability of the derived model. To increase the relevance of the lab experience, the learner will apply the derived model in problem-solving that correspond with the formulated algebraic structure. Sought variables will have their magnitudes within the range of the lab constraints so that the learner can reflect on the conducted real lab while verifying the correctness. Once these two distinct verification processes are finalized, the formulated model will be ready to be confirmed and deployed to other, similar contexts, outside of the activity (e.g., physics, biology, chemistry, or economics), or applied further in engineering designs.

During the STEM modeling activities, a system under investigation as well as its variables must allow being defined using mathematical rules. To increase understanding the meaning of mathematics concepts, students should be encouraged to view a given phenomenon through patterns and rules of mathematics. Suitable standards and their corresponding mathematical embodiments are identified through observing the system behavior, identifying related variables, formulating patterns, and constructing a symbolic representation of the trends. Using derived models in other contexts will constitute the final phase of the activity called *Problem-Solving* or *Engineering Designs*. This phase can be considered a further verification of the derived pattern, and it can be conducted through similar problem-solving related to the examined STEM context. This phase has another purpose; it is to serve the learner as a source of prompts stored in their long-term memories (see Sect. 3.4). This phase is also to increase the students' awareness of the scientific nature of the derived representations and prioritize the interpretation of the formulated algebraic function over the applied procedures. The STEM modeling cycle also proposes a revision process. The stage of revision depends on the model and the degree of its fit to phenomena parameters. Thus, it can begin from revising *Testing and Analysis* or from any stage that the modeler identifies as falsely stated.

6.2 Teacher's Guidance During Multidisciplinary STEM Activities

Although modeling activities are classified as student centered, teachers play a vital role not only in designing but also during the lab conducts. Diefes-Dux et al. (2012) suggested that instructors are partners of innovation during modeling processes.

They should suggest specific approaches, called *corrective guidance*, and help students revise specific modeling processes if the processes do not lead to the desired model formulation. To provide more valuable support, the instructor must be familiar with the concepts of mathematics and science involved in the activity. The instructor might circulate between the students' desks and check the progress by listing to students' discussions and offer suggestions if needed. Students always appreciate any suggestions although they might not always be ready to request them. It is recommended that the instructor conduct activity on his/her own prior is assigning it to the class. Such proactive approach would give him/her opportunities to learn where questions can arise and be ready to answer them. Mason et al. (2009) stated that teachers need to possess strategies and tactics for extracting structural relationships and bring them to the fore for the students. Thus, suggesting students other venues rather than telling students fixed solutions is a priority.

6.3 Sequencing STEM Modeling Activities Within Mathematics and Science Curricula

One of the themes that emerged from the STEM literature was the sequencing of modeling activities within the school curriculum. There can be two distinct voices identified in this debate; one advocated by, for instance, Blum et al. (2007) suggesting that modeling activities be implemented prior teaching new content, and another view presented by Chinnappan (2010) suggesting that modeling activities be applied after new content is delivered to students. Both strategies have individual merits. The decision which sequencing to take depends on a cognitive load of the disciplines and the purpose of the activities. If the cognitive load of the mathematical concept is low and the underpinned scientific context challenging to visualize (e.g., optimizing area enclosed by a string of fixed length) then such activity can be conducted prior introducing the formal algebraic technique of that is embedded in the lab (in here optimization). By using this rote, the students will get familiar with the mechanics of how the areas are being generated. Thus, they will get familiar with the contextual background of the process. This familiarity should help them with an understanding of a formal algebraic technique that will be developed after the lab. While learning the formal technique and solving similar textbook problems, students would have a real representation of the idea coded in their memories that will guide them along necessary trajectories and help to understand the formal technique. Another possible sequencing is conducting a STEM activity after theoretical underpinnings are being delivered to students. This strategy can be implemented when the mathematical representations applied in the activity require new ideas to be learned (e.g., the idea of periodic or parametric functions). Lesh and Kelly (2000) contended that students must possess necessary mathematical tools and knowledge before engaging in modeling activities. Similarly, Koeppen et al. (2008) advocated that students' pre-domain knowledge strongly correlate with their achievement in problem-solving modeling activities. Three of the proposed labs (Chaps. 8, 9, and 10) are suggested to be conducted after the math content is being developed and one activity (Chap. 7) is suggested to be sequenced prior developing a formal mathematical background. Further research that would allow to quantify learning effects of each type of sequencing shed more light on the approaches efficiency. The underlying goal is to have students feel comfortable, enjoy merging the disciplines while constructing new knowledge, and have the students look forward to doing more such activities so that STEM becomes a field of their professional interest.

6.4 General Description of the Proposed STEM Activities

The proposed activities are to serve as contexts for developing students STEM reasoning skills. The time allocated for students to complete each of the activities is about 60 min. Merging mathematics and scientific reasoning is not an easy task, therefore offering more time in class, if possible, is recommended. These activities are designed to be inserted in mathematics curriculum (pre-calculus or calculus) or conducted during separately organized units such as extra curriculum activities. When organized as separate instructional units, e.g., after school, the learning effects of the lab conducts might be higher because the students would extend the time to discuss the lab tasks and thus put more intuition in the lab completion. In all the proposed activities, the final product is a mathematical representation that the learners will derive and contextualize. The selection of algebraic representations or scientific concepts was supported by research in mathematics education that precedes each of the activity. Modeling takes a central stage during these activities, and it is to be perceived as a method of supporting STEM reasoning. Modeling is also to be perceived as an attempt of shifting the learner's focus from deductively searching for a solution to inductively developing concise and general structures. It is assumed that through these modeling activities, students will be made aware that the solutions to problems follow directly from a mathematical model of the problem which appears as a core idea during the investigations. The modeling processes should also serve students as a set of representations stored in their long-term memory that they can retrieve while solving textbook problems. The complexity of the explorations depends on students' background, yet in most of the activities, the necessary mathematical apparatus needs to be learned prior the activity conduct because the purpose of these enterprises is to merge known mathematics structures with science. For instance, if a quadratic function (Chap. 10) is to be applied and interpreted, students need to recognize its property to take a maximum or a minimum value. If a linear dependence is to be used, one needs to acknowledge that the rate of change between involved variables remains constant (Chap. 8). Without a prior understanding of algebraic structures and their attributes, merging the knowledge might result in endless trials. Research showed that students need to be familiar with scientific contexts of the labs as well. Therefore, the selection of the contexts was followed by analyzing science curricula and assuring possessing that knowledge by students. From the educational research point of view, all the labs can be classified as case studies. While a detailed quantitative pre-posttest analysis was not employed, students' pre-posttest or posttest verbal responses were collected, analyzed, and discussed. It is hoped that even the general evaluation can serve as a departing stage for applying and expanding these projects in any school setting.

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Part II
Scientific and Mathematics Reasoning in STEM Practice

Chapter 7
Modeling with Exponential Decay Function

Abstract Understanding rates of change sets the foundations for differential equations that are central to modeling dynamic phenomena in science and engineering. While modeling with a constant rate of change is well understood, modeling an exponential change requires a more detailed approach due to a diversity of its computing. An exponential model can be characterized by a constant ratio of change of the quantity, a constant percent rate, decay or growth factor, decay or growth rate, and so forth. The purpose of this STEM activity was to create a STEM environment that would enable the learners to discover a constant ratio as a departing block to construct an exponential model. An experiment of investigating maximum heights of a bouncing ball was used as a scientific context. This phenomenon was selected due to providing observable and measurable data. A pretest analysis was used to identify students' weaknesses, and the posttest analysis has been used to discuss students' changes of perception on modeling exponential behavior. The STEM context has disclosed that the commonly applied term *decay* denoting a decreasing exponential function did not adequately describe the virtue of decaying processes. Conclusions and suggestions for further studies are discussed.

7.1 Prior Research

Exponential behavior is an essential part of any mathematics curriculum. Its applications are widespread in the biological, physical, and medical sciences. Continuous exponential functions dominate the applications of the exponential model in science, and their discrete counterparts extend the applications to business, finance, economics, and accounting. More specifically, exponential decay is being used in RC (Resistor-Capacitor) circuits, damped harmonic oscillations, the growth of a nuclear chain reaction, attenuation of sound or light, compound interest, population growth, logistic growth, and so forth. The set of applications further expands when differential equations involving constant percent rates of change are considered. What makes the process of formulating an exponential function distinct from, for example, linear? Thompson (2011) claimed that one of the most critical aspects of the exponential model is understanding the idea of *rate* as being able to attend to

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investigating changing phenomena outputs and simultaneously compute the rate of the change of that outputs over a specific domain. These deficiencies draw on limited opportunities that math students experience with exploring dynamic phenomena. In the line of this finding, several educators emphasized high effects of contexts in developing the understanding of functions and rates. When considering an exponential decrease, a radioactive decay seemed to be the most prevailed that students experience in high school education. To reflect on applying a dynamic phenomenon, Brendefur et al. (2014) used a setting of bacterium population to spur students' free thinking about exponential growth. Jesse (2003) suggested an activity that utilized a series of time intervals and plotted the amount of mass as a function of the accumulated time on a semi-logarithmic scale on which the slope of resulting linear graph represented the time of the half-life of the substance. While deciding about using an exponential function, Huestis (2002) suggested that an exponential decay function is used because of two physical principles: (a) radioactive nucleus has no memory and (b) decay time interval for any two nuclei of the same isotope are governed by the same probability distribution. Castillo-Garsow (2013) focused on contrasting continuous and discrete growth rates with the aim of finding similarities between these representations. Touchstone (2014) proposed calling exponential *decay* a *survival* and expressed it as \$1$(1 - r)$ and *growth* expressed as \$1$(1 + r)$ called *arrival*. The symbol r in both models represented the magnitude of growth or decay rate, respectively. Lo and Kratky (2012) found out that students had difficulties determining whether a given real-life situation would be best modeled by a linear or an exponential function. They claimed that this was accounted for a lack of understanding of the nature of rate of change in each of the models and further suggested to emphasize that in a linear model, $y = mx$, the rate, m is constant, and that the rate can take different values for an exponential model, $y = a(b)^x$, depending on the value of the base. Wanko (2005) suggested applying the idea of probability and using a graphing calculator to find an exponential regression and identify the decay/growth rate. Ärlebäck et al. (2013) proposed measuring the potential difference on a discharging capacitor to model exponential decay and found out that students' difficulties projected from either the interpretation of the decimal value as a percentage of the voltage change across the capacitor, or from being unable to identify and interpret the time factor in the interpretation of the decay constant. By selecting a context that required an extensive physics background, these researchers realized also that students who attempted to interpret the constant base using the context of the model were less successful than those who attended to only the mathematical aspects of the constant. Developing students' ability to correctly interpret percentage which arises from the constancy of the successive ratios in an exponential model was flagged by Ärlebäck et al. (2013) as an area in need of further research. Shea (2001) investigated how science textbooks introduce the idea of exponential behavior and found out that about 90% of 60 science textbooks poorly elaborated on the algebraic underpinnings of the concept of decay. A typical introduction of exponential growth in a high school mathematics courses, according to Confrey (1991), was to post a problem that required repeated multiplication and then connect that algorithm to an exponential notation that enabled the students to construct an exponential function. Such methods of introducing exponential behavior prevail in currently used math textbooks as well. A post hoc analysis of some pre-calculus textbooks (e.g., Stewart 2006; Swokowski and Cole 2008) has revealed that modeling with exponential functions is typically initiated by successive multiplications of the base that lead to an exponential algorithm. Subsequent word problems provide students often with a formulated decay or growth function and require using a calculator to evaluate the models. It seems that working on such problems, does not expose the students to the process of building up the model and diminishes the task of critically evaluating given data for the exponential model to be applied.

In sum, the review shows that while science emphasizes the interpretation of exponential behavior, mathematics emphasizes the algorithm for computations. A lack of underlying the connections can be a reason for students' difficulties in problem-solving that require merging these two views. Furthermore, there is little agreement on how the exponential growth or decay should be introduced and how to overcome challenges in a manner consistent with both science and mathematics methods. The decay or growth factors are often extracted from stochastic processes or by an iterative multiplication or division processes. These extractions are often made without a clear connection to their corresponding percentage interpretation that is very often used in everyday life to describe exponential changes. It is hypothesized that the lack of connections causes that the exponential behavior is challenging for students to express as an algebraic function. The literature provides numerous studies where the idea of exponential behavior was investigated. However, STEM studies integrating scientific and mathematical interpretations of the model have not been found. The STEM activity is an attempt to fill in the gap.

7.2 Analysis of Pretest Findings

The prior research revealed a high diversity of the methods of mathematizing exponential behavior and a vast range of applications. Both these findings signify the importance of these topics in students' general STEM disposition. To learn more about students' perceptions of the structure of exponential functions, I have administered a pretest to a group of 25 pre-calculus students. These students had studied exponential functions in previous math classes. The pretest focused on the core idea of an exponential behavior, thus the nature of the base of the model. Students were to provide a verbal response to the following question: *What is the interpretation of the base of an exponential function when applied in real contexts?* The students' responses were clustered into three groups due to their quality. Responses placed in group 1 ($N = 4$, 16%) contained the term *percent* or *rate* to describe the base. Responses to group 2, with the largest population ($N = 15$, 60%) linked the base with the initial value of the quantity. Group 3 ($N = 6$, 24%) contained responses that did not contain any terms that would indicate an understanding of the base, for example, one student wrote: "*the base shows the lowest possible denominator*."

Table 7.1 Findings of students' interpretations of exponential models

Challenges with understanding of exponential model	Suggested modifications
A lack of clarity of how the base of the exponential model is computed.	Introducing the ratio as a fundamental building block of an exponential function.
A weak link between the exponential model and the idea of the percent.	Introducing percent as an imminent concept derived from the ratio.
An unclear position of the initial value of the quantity in the exponential model.	Underlying the quantities initial value as a component indicating the physical unit of the exponential model.
Lack of differentiating between the amounts that decay and the amount that remains within a system.	Emphasizing the base as a ratio indicating the percent of the quantities that remain within the system.
Lack of opportunities to verify formulated the model in real-life applications.	Using experimental setup to have students verify a coherence of the derived model with measurable outcomes.

The responses revealed that the students understanding of formulating the base in the exponential model is weak. It was also interesting to note that not many students associated constant percent ratio of quantities of interest as a building block of the exponential model. It could be further concluded that students did not have a clear picture on how to differentiate between rate and ratio which could also transfer to difficulties with differentiating between linear and exponential models that were also reported by Ärlebäck et al. (2013). A large group of students mistakenly associated the base of an exponential function with the initial function value or its output value. These students attributed the meaning of *the base* as perhaps *a base function value* that provided a starting point for the decay or growth. Thus, a necessity to differentiate between these function components emerged, and a STEM activity whose independent and dependent variables can be conceptualized using their physical units provided an excellent environment for attempting to establish clarity of these function components.

Table 7.1 summarizes findings from prior research and the pretest about students' handling of exponential models. The table contains also proposed modifications of introducing the model that was pursued in this study and included in the instructional support of the lab.

Discussion of these modifications is presented in Sect. 7.3.

7.3 Introducing Exponential Model Using STEM Contexts

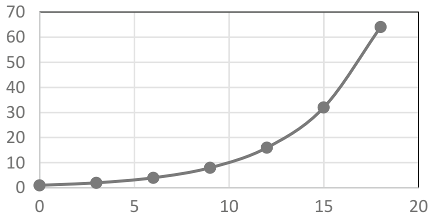
This section provides a theoretical background that should familiarize the instructor with the activity design and its STEM intertwining. The focus of this part is contrasting rates and ratios as students typically study the ideas in their primary mathematics education. Bringing forth these concepts and highlighting their similarities and differences prior discussing exponential behavior appeared as a connection worthy of establishing. Departing from that comparison, the contextual introduction of the exponential model is also discussed.

7.3.1 The Difference Between Rates and Ratios

What is the difference between ratio and rate? Arons (1997) defines ratio as a division of two homogeneous quantities, for example, $\frac{12\,\mathrm{m}^3}{6\,\mathrm{m}^2}$, $\frac{3\,\mathrm{cm}^2}{2\,\mathrm{m}^2}$, or $\frac{3\,\mathrm{cm}^2}{8\,\mathrm{cm}^2}$. Rate is defined as a division of two heterogeneous quantities, in which the denominator represents usually the quantity of time, and the numerator, the quantity of interest, for example, $\frac{7\,\mathrm{m}^2}{6\,\mathrm{s}}$, $\frac{20\,\mathrm{kg}}{5\,\mathrm{m}^2}$, $\frac{3\,\mathrm{m}^3}{6\,\mathrm{h}}$. Ratios and rates can be reduced or simplified by division. After that operation, the resulted magnitude represents a *unit ratio* or a *unit rate*, for example, $\frac{12\,\mathrm{m}^2}{6\,\mathrm{m}^2} = 2\frac{\mathrm{m}^2}{\mathrm{m}^2}$, $\frac{20\,\mathrm{kg}}{5\,\mathrm{m}^2} = 4\frac{\mathrm{kg}}{\mathrm{m}^2}$. Ratios can reduce to be dimensionless however, rates retain the units. Even though rates can be dimensionless, it is recommended that the units are kept with the magnitude for their correct interpretation, if not involved in a percentage conversion. Both representations can lead to formulating *proportions* under the conditions that both are embedded in *direct proportionality* relations. For example, $\frac{20\,\mathrm{kg}}{5\,\mathrm{m}^2} = \frac{x}{2\,\mathrm{m}^2}$ or $\frac{12\,\mathrm{m}^2}{5\,\mathrm{m}^2} = \frac{8\,\mathrm{m}^2}{y}$. If the left side of the proportion contains two variables, a linear function can be formulated; $\frac{20\,\mathrm{kg}}{5\,\mathrm{m}^2} = \frac{y}{x}$ thus $y = \left(4\frac{\mathrm{kg}}{\mathrm{m}^2}\right)x$, where in this case, y represents kg, an x represents m^2. The rate $4\frac{\mathrm{kg}}{\mathrm{m}^2}$ takes another more sophisticated interpretation in this model *as the slope* when it is graphed in the Cartesian plane. Visually, the slope illustrates the inclination of the line to the horizontal axis, commonly known as $\frac{\text{rise}}{\text{run}}$. If a set of variables is considered, a constant rate of change of two quantities implies the application of a linear function.

7.3.2 Building Contextual Background to Elicit an Exponential Model

A ratio of a dimensionless quantity can lead to expressing the quantities in a percent form. For example, if a colony of bacteria increased from 16 to 32 in 3 h, then the ratio of the populations can be expressed as $R = \frac{32}{16} = 2$. One can say that the population increased twice or by 100% of its initial value over the period of 3 h. If more data can be collected (see Fig. 7.1) and if that ratio proves to remain constant over each 3 h period, then an exponential function can be constructed. How to make a transition from a ratio representation to a function? Let's generate a table of values and plot the function graph (see Fig. 7.1).



Time (hours)	Bacteria population (in hundredth)
0	1
3	2
6	4
9	8
12	16
15	32
18	64

Fig. 7.1 Graph and table of values representing an exponential growth

Let's check the behavior of the rate of change of population versus time and examine if a linear model can be used. By verifying three rates, one learns that:

(a) $\text{For}\, 0 \leq t \leq 3\,\text{h}, \text{rate} = \frac{100}{3}\frac{\text{bacteria}}{\text{hour}}$

(b) $\text{For}\, 3 \leq t \leq 6\,\text{h}, \text{rate} = \frac{200}{3}\frac{\text{bacteria}}{\text{hour}}$

(c) $\text{For}\, 6 \leq t \leq 9\,\text{h}, \text{rate} = \frac{400}{3}\frac{\text{bacteria}}{\text{hour}}$

Thus, the rate of change of the bacteria is not constant, and a linear model must be rejected from the consideration. Let's compute the ratios, labeled R, of the populations after each 3 h period:

(a) $R = \frac{P(3)}{P(0)} = \frac{200 \text{ bacteria}}{100 \text{ bacteria}} = 2$

(b) $R = \frac{400}{200} = \frac{4 \text{ bacteria}}{2 \text{ bacteria}} = 2$

(c) $R = \frac{800}{400} = \frac{8 \text{ bacteria}}{4 \text{ bacteria}} = 2$

The physical function units, the number of bacteria cancels out producing dimensionless ratios of function values that are equal to 2. This constant ratio implies an exponential behavior and thus an exponential model to be applied. Thus, for an exponential model to be valid, one must seek a *constant ratio*; $R = \frac{y_n}{y_{n-1}}$ that must be constant for any two consecutive values. Other interpretation of that statement is that the quantity change is proportional to its current value $y_n = (y_{n-1})R$. In the discussed above example, the population of the bacteria culture, let's call it $P(t)$, doubled every 3 h or increased 100% of the prior population after every 3 h, thus $P(t_2) = P(t_1)$ (2) which leads to $P(t) = P_o(2)^{\frac{t}{3}}$ if a function representation is considered. The modification of the expression for the function exponent is dictated by the growth factor that was computed over a period of 3 h. In that model, P_o represents the initial value of the quantity, (the population), and 2 is the ratio of any two consecutive function values that must always be dimensionless because it is used to compute the magnitude of the function value not its physical units. This statement can be generalized as

$$\text{Function value at } (t) = (\text{The initial function value})\,(\text{Ratio})^{\frac{\text{General time variable}}{\text{Time period of the ratio computing}}}.$$

Symbolically, this expression takes the form: $P(t) = P_o(R)^{\frac{t}{T}}$. It is to emphasize that the physical unit of the quantity of interest is determined by the unit of the initial function value. While not discussed in mathematics textbooks, the exponent must also be dimensionless. If in $P(t) = P_o(R)^t$ the time, *t,* is expressed in years, then in the full form of the function is $P(t) = P_o(R)^{\frac{t}{1\,\text{year}}}$. In the textbooks, such function is rather expressed as $P(t) = P_o(R)^t$. When context is provided, the significance of the dimensionless form of the ratio R, and that of exponent can be brought to students' attention. These two elements can be used as additional means of verifying that the model is correct. How to compute the percent of decay or growth?

If $R > 1$, for example, $R = 1.2$ then a growth function can be constructed. In this model, 0.2 is called the growth factor and the percent growth in this model is 20%. It also means that the system accumulates 120% of its initial amount over a specified time period. The interpretation of the ratio, for instance, $R = 0.7$ as a decay factor requires a more detailed insight.

If $0 < R < 1$, for example, $R = 0.7$, then it means that 30% of the system value *decays* and the system retains 70% of its previous value. The lab has revealed that students often think that the decay function computes the amount of the quantity that decays, not the one that the system retains. Labeling the value of 0.7 a decay factor, because of its value less than one, does not explicitly reflect on the system direction of change, and students need to be explicitly informed about this nuance. In sum, in both models, the ratio represents a division of $\frac{y(n+1)}{y(n)}$ of function values that shows as a ratio of *retained quantity* within the system (or experiment), not the ratio of function values that decays. This delicate nuance can be discussed with students while performing real experiments when the context can be used to support the reasoning. It is to note that a typical radioactive decay with a base of ½ is not sensitive to this nuance.

7.3.3 Logistics of the Lab

The instructor explained the process of taking data and highlighted the precision of data taking. The instructor explained that while the students would draw a continuous graph, a better fit for the data was a discrete form. A continuous model could also be considered to visualize the ball's maximum height. A class of 25 students worked in groups of 3–4 students. Students took data in groups and discussed the lab completion in groups. However, each student was supposed to have own hypothesis and computations. Each group received a baseball and a meter stick. One student of each group dropped the ball from a 2 m height and the other recorded its movement using a video camera. Videotaping allowed for more precise measurements of the successive heights. Each student received an instructional support (see the Sect. 7.4).

There was a physics component added at the end of the lab analysis; the law of conservation of energy that provided a bridge to problem-solving in science. Students, especially those taking a physics course found that part intriguing and thought-provoking.

7.4 Lab Outline

Purpose: In this lab, you will formulate an algebraic function to express a maximum height of a bouncing ball. You will also construct a function that would represent the maximum potential energy of the ball as it bounces and apply this idea to formulate a general law of conservation of energy for the bouncing ball.

Materials Meter stick, a variety of tennis balls, French curves, TI-84, iPhone to videotape the motion.

Problem 1:What type of function, exponential, linear or quadratic, etc. can be used to model the maximum height of a bouncing tennis ball after each rebound? Justify the answer.

Hypothesis ____

Problem 2: What quantities are needed to construct the function? Support your answer.

Prediction____

Problem 3: If the graph is plotted in height vs. bounce number axes**,** will the graph have:

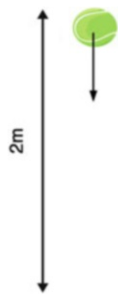
(a) A real y-intercept?

Prediction____

(b) A real x-intercept?

Prediction____

7.4.1 Lab Procedure

You will work in groups of three students and videotape the motion of the ball to attain more precise measurements.	Meter stick
• Release the ball from a height of 2 m and measure the maximum height to which the ball raises after the first, second, third, and fourth bounce. *Note videotaping will help to identify the height more accurately.*	
• Record the heights in the table provided below and each value to the nearest centimeter (e.g., 1.51 m, or 0.72 m).	
• Note *n,* in the table, represents the bounce order number, e.g., for $n = 1$, you will record the height after the first bounce.	
• Perform two separate trials and find the average heights for each bounce.	

7.4.2 Data Analysis

On the grid paper provided below, sketch graph of ***Average height after each bounce*** **versus** ***bounce number.*** Use the highlighted rubrics from the table (the last row from Table 7.2).

- Identify the independent variable (IV) and label it on the horizontal axis ______
- Identify the dependent variable (DV) and label it the vertical axis___________
- The initial height of 2 m represents the y-intercept of the function and thus its initial value.
- Spread out the horizontal scale so that the graph fills in the entire grid (e.g., begin from 0 and count six grids to label $n = 1$, etc.).
- To assure correct labeling, label *Average Height* on the vertical axis and *Bounce Number* on the horizontal axis.
- Use *French curves* to draw a smooth graph (Fig. 7.2).

Now you will formulate an algebraic model for the graph. Read and answer the questions that follow. The answers will guide you through the process of finding the model.

- What type of function does the graph resemble? ________________________

Table 7.2 Maximum height of a bouncing ball after each consecutive bounce

	Initial height $n = 0$	Height after $n = 1$	Height after $n = 2$	Height after $n = 3$	Height after $n = 4$
Trial #1	2 m				
Trial #2	2 m				
Average height (m)	2 m				

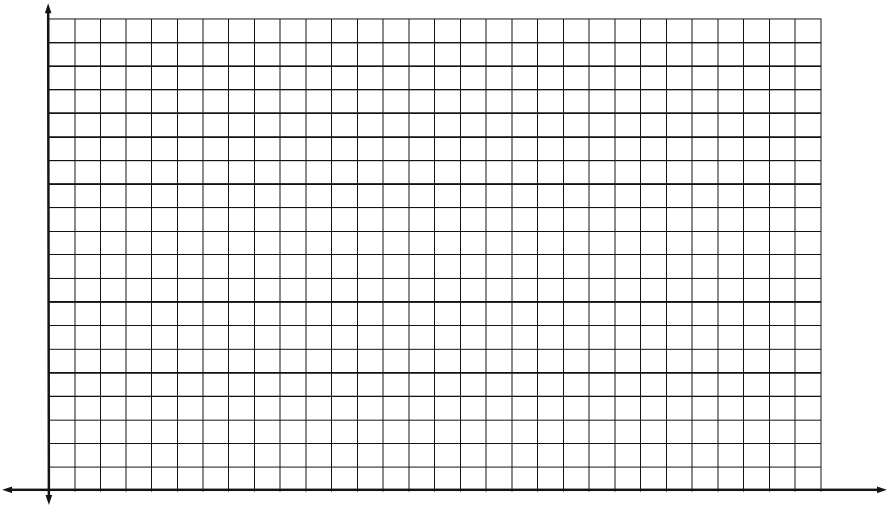


Fig. 7.2 Graph of height vs. bounce number

Table 7.3 Computations of the base of the exponential model

Formula	Ratio (R)
Decay ratio at the first bounce $= \left(\frac{\text{Averge height after the first bounce}}{\text{The initial height}}\right)$	
Decay ratio at the second bounce $= \left(\frac{\text{Averge height after the second bounce}}{\text{Averge height after the first bounce}}\right)$	
Decay ratio at the third bounce $= \left(\frac{\text{Averge height after the third bounce}}{\text{Averge height after the second bounce}}\right)$	
Decay ratio at the fourth bounce $= \left(\frac{\text{Averge height after the fourth bounce}}{\text{Averge height after the third bounce}}\right)$	

- Do you expect that the graph will ever cross the horizontal axis? Support your answer using the experiment outcomes. ______________________
- What is the physical unit of the base? What is the interpretation of the base of exponential function? Provide an explanation using the experiment context.

  ____

- The main component of any exponential function is the base of the power of the function. This experiment is represented by the ratio of two consecutive heights of the ball. Should the ratio remain constant after each bounce? Support your answer? ____
- What parameters affect the magnitude of the ratio? Select the once that you think apply: mass of the ball, properties of the floor, size of the ball, acceleration due to gravity, the initial height of the ball, the air resistance, others: ______________
- You will calculate the ratio for two consecutive bounces and take the average to increase the precision. Express the magnitude (the value of the ratio) to one decimal place and record your computations in Table 7.3. The value of the ratio $0 < R < 1$ represents *a decay*. However, it represents ratio of the heights that the system *retains*. Refer to Table 7.2 to compute the average retaining ratios.

Table 7.4 Summary of the computed individual bases

Bounce number (n)	1	2	3	4
Decay ratio (R)				

- Summarize the computations in Table 7.4.
- Does the *Decay* ratio appear to be constant? (Discrepancies within 10% can be accounted for a low precision) ______________________________
- What is the average value of the decay ratio? $R =$ ______________________
- Does the ratio retain a physical unit? ______________________________

Referring to the value of the decay ratio answer the following:

(a) What is the decay factor in this model? ___________________________
(b) What is the percent of the height lost by the ball at each bounce? __________
(c) What is the percent of the height retained after each bounce? ____________

The decay ratio R provides the base of the power of the exponential model. To formulate the model, the initial value of the quantity of interest (the ball's initial height above the floor) is needed. Note that the unit of the initial quantity provides also the physical unit of the function values. Identify the initial height and complete the model.

- Write the function equation referring to the general model $H(n) = H_o(R)^n$. _____
- Verify your hypotheses and predictions to the problems #1–3 from the first page of the lab and comment any discrepancies.

Hypothesis 1 ____
Hypothesis 2 ____
Hypothesis 3 ____

7.4.3 Model Verifications

1. Suppose that you dropped the ball from a height of 1.2 m.
   (a) Using the function $H(n)$ that you derived, compute the height to which the ball should rebound. ______________________________________
   (b) Verify the computations by dropping the ball from the indicated height, 1.2 m and measure the height to which it rebounded. ____________________
   (c) Do the values correspond? If not, provide a source of error.
2. Do you expect that the value of the base of the exponential model R depend on the initial height of the ball? Support your answer. ______________________
   (a) Drop the tennis ball from a height of 1.7 m and record the height to which it rebounded____
   (b) Calculate the ratio of the heights. _____________________________
   (c) Does the value of the ratio correspond with the one earlier computed?

(d) Compute and interpret the following limit;
$\lim_{n \to \infty} H(n) =$________________
(e) Does the limit value support the experimental outputs? ______________

3. Now you will use a graphing calculator and generate the algebraic model for the function following general procedures (refer to the TI instructions) of finding a regression curve.
Report the general model $H(n) =$ ___________________________________

4. You have learned that the initial function value (the initial height) has no bearing on the decay ratio (and consequently the percent rate). Provide possible explanations for that independence.____

7.4.4 Integrating a Physics Component

Part 1: Finding function to model gravitational potential energy of the bouncing ball.

In this part, you will merge further a scientific context that pertains to the computed decay ratio. The following is a contextual background that will help you apply math tools to analyze the situation.

An object that is held above a certain height will possess a gravitational potential energy that can be computed using $U(H) = mgH$ where m is the mass of the object expressed in kilograms, g is the intensity of gravitational filed on the earth; $g = 9.8$ m/s^2, and H the height expressed in m. Note that $U(H) = mgH$ can be considered a linear function due to H being the only variable. If the height H and the mass is known, the function can be used to compute the energy. Suppose though that you are to compute the gravitational potential energy $U(H)$ of the ball after each *bounce*. How to construct a $U(H)$ that instead of H will have n as the independent variable? Answering the questions that follow should help you formulate the $U(n)$.

1. Can $H(n)$ represent an inner function in (n)? ___________________________
2. Write down $H(n)$ that you have formulated in the Sect. 7.4.2
3. Formulate $U(H(n))$=____
4. On the grid provided below, sketch $U(H(n))$.

Note the independent variable of the $U(H(n))$ is the bounce number n.

Part 2: Finding function to model energy dissipated by the ball at the impact.
The collision with the floor is non-elastic because the ball loses gravitational potential energy shown by a decrease of height to which it bounces up. The loss of potential energy is converted into thermal energy (heat) denoted usually as Q. How to find a function to model the heat dissipation? Finding empirically the function representing the heat dissipated is not an easy task because there are too many factors affecting this process. Thus, other means need to be developed. One of

the means is considering the ball-floor as a closed system that retains its total energy throughout the entire experiment. What is the amount of the energy? You will note that the energy of the system is equal to the initial energy of the ball at the height of $H = 2$ m. Let's denote it $U_i(2)$. Compute the value of the energy using the experiment outcomes

$$U_i(2) = mg2 = \text{------------------------------------}$$

As the ball bounces up and down, where n represents an nth bounce, the total energy of *the ball-floor system* can be expressed as $U_i(2) = U(n) + Q(n)$ and it remains constant due to the system being isolated and applied law of conservation of energy. As the gravitational potential energy of the ball decreases, the heat energy increases due to an inelastic collision of the ball with the floor. The goal of the reasoning is to learn how the $Q(n)$ function behaves. Following are some steps that should help you with the process.

- Replace $U_i(2)$ by the computed earlier value and solve the formula $U_i(2) = U(n) + Q(n)$ for $Q(n)$=____________________________________
- To sketch $Q(n)$, find its initial value, thus $Q(0) =$ and the limit $\lim_{n\to\infty} Q(n) =$ ______
- What is the physical interpretation of the limit of the heat function? __________
- Sketch the function on the same axes (Fig. 7.3).
- Compare the limit values of $\lim\limits_{n\to\infty} U(n) =$______ $\lim\limits_{n\to\infty} Q(n) =$________________
- Do the limit values reflect the reality of the experiment? ____________________

7.5 Lab Reflections

What have you learned? What? ____
____

7.6 Posttest Analysis and General Discussion

Students took the data and formulated the exponential equation without major obstacles.

Three problems that students answered the next day served as a posttest measure. In addition to a parallel pretest question, the students were to solve four quantitative multiple-choice problems whose evaluation follows. These problems focused on testing the students' ability to identify a correct model for an exponential growth or decay. Percentages of correct answers are shown in bold font.

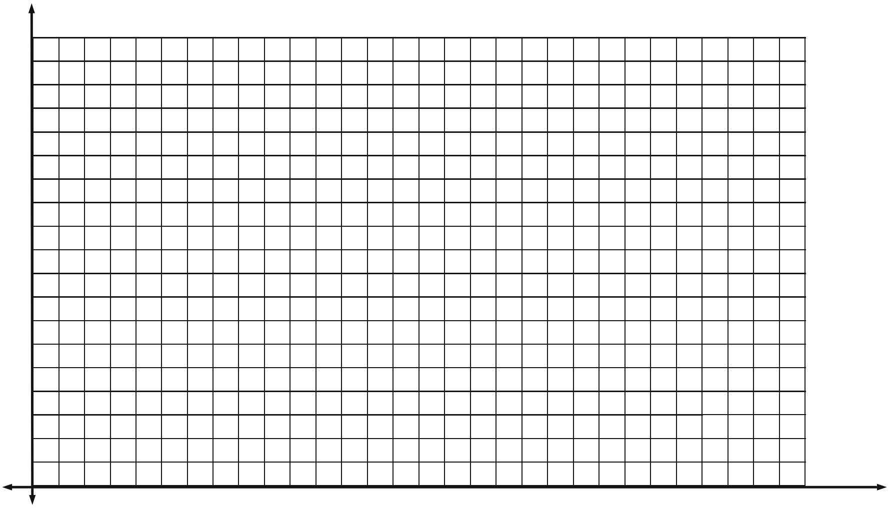


Fig. 7.3 Graph of the potential energy of the bouncing ball and the maximum height

Problem 1: A sample of a radioactive substance has an initially a mass of 300 g. After one day, the sample retained 6/7 of the original mass. Which function best models the mass of the sample, $m(t)$, at any time, t?

Answer choices	Students' choices (%)
$(A)\ m(t) = 300(7/6)^t$	6
$(B)\ m(t) = 300(1 - 7/6)^t$	43
$(C)\ m(t) = 300(6/7)^t$	**51**
$(D)\ m(t) = 300(7/6 - 1)^t$	0

This problem tested students' ability to identify a radioactive decay model. While many of the students selected a correct answer (C), surprisingly there was a high percentage who selected choice B. This could be accounted for an association of that form to a decay model $(1 - r)$. These students did not realize that this choice is false because the base takes a negative value which is not acceptable in any exponential function. To improve the percentage of correct answers on this question, an emphasis that the base must represent the ratio of consecutive function values is one of the options.

Problem 2: A tennis ball was dropped from a height of 3 m The ball lost 20% of its original height after each bounce. Which of the functions best models the height $h(n)$ of the ball? The variable n represents a bounce number.

Answer choices	Students' answers (%)
$(A)\ h(n) = 3(1 + 0.2)^t$	0
$(B)\ h(n) = 3(0.8)^t$	**80**
$(C)\ h(t) = 300(1 - 0.8)^t$	9
$(D)\ h(t) = 300(-0.2)^t$	11

This problem was rooted in the context of the lab, and its high correct percent rate can be attributed for students' referencing the provided STEM context. Though, about 20% of the students misinterpreted the algebraic meaning of the base when percentage was given. The idea of building an exponential model given by a percent rate was embedded in the lab, yet there seems to be more done to differentiate between given decay percent rate, and its relation to interpretation of the ratio of the decay model.

Problem 3: The initial population of a colony of bees was 100. It has revealed that the ratio of the population computed over a 1 h period was four. Which function best models the population, $P(t)$, of the colony in the function of time, t?

Answer choices	Students' answers (%)
$(A)\ P(t) = 100(4)^t$	**69**
$(B)\ P(t) = 100 + 4t$	23
$(C)\ P(t) = 100(4)^t$	4
$(D)\ (t) = 4(100)t$	4

This problem targeted the students' ability to differentiate between rate and ratio and respectively a linear and exponential model. Majority of the students considered the given ratio as a dimensionless quantity representing the base of the growth model and selected a choice C. Some students ($N = 6$, 23%) considered the ratio as a rate of a linear model. While there was no direct information in the problem, about *exponential* growth of the population, a linear model could have been rejected by verifying the physical unit of the quantity that resulted from considering a linear model; $[P(t)]$ = [number of bees + time unit] that leads to a clear inconsistency in formulating the population function. Verifying units as a support for formula confirmation is used in science, mainly physics. Inclusion of this technique to support problem-solving in mathematics seems as one of the areas increasing STEM readiness.

Problem 4: At the time 1 PM, the number of bacteria in the culture was 200, at the time 3 PM, the culture grew to 800. What is the growth ratio of the bacteria culture?

Answer choices	Students' answers (%)
(A) 1/4	12
(B) 1/3	4
(C) 3	36
(D) 4	**48**

This problem tested students' ability extracting necessary information to build the base of exponential model. The growth ratio was computed correctly by 46% of the students. A high percentage of students (36%) selected an answer that resulted by creating a quotient of the time instants. These students computed $\frac{t(n+1)}{t(n)}$ instead of $\frac{y(n+1)}{y(n)}$. While both ratios did not carry out dimensions, ratio of time instants is not a part of the exponential model.

When considering differentiation between exponential decay and growth (across all these problems) the students selected correct choices. It seemed though that more attention is needed to differentiate between rates and ratios as using either linear or exponential model follows this differentiation. Students would be encouraged to adopt the technique of verifying the unit of the computed quantity as an additional means to support algebraic processes.

To have students acquire a deeper understanding of these concepts, providing a general categorization of the types of problems on exponential modeling is recommended. I suggest formulating three general categories depending on what quantities are provided (see the categories below). Each set of provided quantities will lead to finding a general exponential model. Such generalization can be discussed with students using the lab outcomes or other contexts.

Category 1: Provided is a set of two pairs of coordinates. For example, (2,*f*(2)), (6,*f*(6)), then compute the ratio, $R = \frac{f(6)}{f(2)}$. Identify the *period* for computing the outputs: $T = 6 - 2 = 4$ and use$f(t) = f(2)(R)^{\frac{t}{4}}$. This model can also be derived using system of two exponential equations.

Category 2: Provided are the initial value P_o and a percent growth ratio computed fixed time interval. For example, $R = 2\%$computed over 3 h. Use the same general model, however pay more attention to the base formulation that is $R = 1 + 0.2$ and $P(t) = P_o(1.2)^{\frac{t}{3}}$.

Category 3: Provided is a decay ratio. As earlier discussed a caution need to be given to problems that explicate on the percent left or decayed within the system. For example, if 60% of the quantity decayed over 5 h, then the amount left within the system can be computed from $(t) = f(0)(0.4)^{\frac{t}{5}}$. According to the general interpretation, the ratio represents what part of the initial value is *retained* in the system.

The students should be provided with opportunities to make transitions from one category to another.

The law of conservation of energy (physics addition to the lab) was not included on the posttest. This part of the lab despite provided contextual support appeared to be more challenging than expected. Many students tried to find the heat function Q by referring to the standard formula $Q = mc\Delta T$ and could not move forward with the solution process because of the inability to find the mass, heat capacity, and the temperature change of the floor. Upon providing further suggestions on expressing the law of conservation of energy as a function and solving the equation for Q, the students completed that part. Students, especially these taking physics engaged in the math-science interface more deeply than the other. However, it is seen that more scenarios of similar contexts should the students practice prior applying this idea is the real experiments.

The students did have questions regarding the interpretation of the decay factor, decay ratio, and the relation between percent of the quantity that decays and a percent of quantity that the system retains. This area was addressed earlier, however it required more elaboration. Decay according to the dictionary (The American Heritage College Dictionary 1997, p. 358) means to disintegrate or diminish. Thus, this term is associated with the amount of the quantity that exits the system not with the one that remains in the system after the decay process. In mathematics textbooks (e.g., see Stewart 2006, p. 245; Swokowski and Cole 2008), the decay factor is called the base of the exponential model. Following this definition, the decay factor should rather represent the amount of quantity that decays not the quantity that remains within the system. Thus, the question that arises is if labeling the base of an exponential model; a decay factor accurately reflects the physical properties of the modeled quantity. It seems that the decay ratio should represent the fraction of the quantity that disintegrated not the one that remained in the system. For example, if the initial mass of a sample is 300 g, and 100 g of the mass decayed, then the decay ratio should rather be 1/3, not 2/3 because the fraction of 2/3 represents ratio of masses that did not decay. This misinterpretation is even more visible when question asks directly about computing the amount of substance that decayed. In such questions, students mistakenly evaluate the formulated decay model. The STEM activity when the students could observe the quantity change disclosed the nuances and then a need for more research in exponential behavior the terminology used. Students need to be provided with more details about the formal interpretation of the decay model before immersing in the lab conduct and work on word problems.

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Chapter 8
Exploring Function Continuity in Context

Abstract Although function analysis is widely applied in science, there are some areas, like function limits or continuity that are underrepresented. The purpose of this study is to model motion and support this process by applying function continuity. Students will model the principles of continuity by formulating position functions for objects moving along a horizontal path with multiple rules. The scientific context will be supplied by an interactive simulation called *Walking Man* that is available for free at http://phet.colorado.edu/en/simulation/moving-man. This simulation allows designing a movement that can be mathematized using piecewise polynomial functions. Function continuity and sided limits will be used as tools to support the construction of these functions. The activity was conducted with a group of 20 calculus students. It was hypothesized that by applying the principles in context, the students would realize that function continuity is a critical condition that functions representing motion must satisfy. Posttest results supported the hypothesis.

8.1 Introduction

Conditions that warrant function graphs adherence to model motion are following: (a) test function, commonly known as vertical line test, (b) continuity that prevents gaps and breaks in the function values, and (c) differentiability that assures continuity of the derivative function. All of these conditions are essential topics of typical calculus courses. However, applications of these fundamental principles that would support the importance of their inclusion in mathematics curricula are not often presented to students. A preliminary survey of calculus textbooks has revealed that typical problems on these principles gravitate toward verifying if given function is continuous or differentiable at a specific x-value in its domain. Narrowing the assessment problems to context-free questions diminishes the importance of the principle of continuity. Functions representing motion such as position-time graphs, velocity-time graphs, and acceleration-time graphs constitute an integrated part of most calculus courses, yet the condition that warrants the functions continuity is not being discussed. The research on graph construction, sketching, and interpreting concerns equally the science and mathematics communities (Sokolowski 2018).

A. Sokolowski, *Scientific Inquiry in Mathematics - Theory and Practice*,
https://doi.org/10.1007/978-3-319-89524-6_8

Therefore, immersing calculus students in a lab that would allow exploration of continuity in a STEM context appeared worthy of undertaking. In addition to exploring continuity and limits, the students will develop techniques of constructing piecewise functions based on simulated motion.

8.2 Prior Research Findings

Since this study will encompass not only a general function analysis but also the idea of piecewise functions and the concept of limits, a survey of prior research will target literature findings in all these areas. The ability to interpret graphs and learn about the motion, beyond what the graph shows, is essential for general mathematical and science literacy. Research shows that students frequently struggle with these tasks. Smith (1996) found out that students often apply fragmented, memorized rules while solving motion problems. Soon et al. (2011) argued that students' deficiency in that areas is due to their inability to initiate a transfer of mathematical concepts to contexts outside of *mathematics classroom*. One of such difficulty that drew a substantial body of research was the inability of translating between object's path of motion and a respective position-time graph (Leinhardt et al. 1990). In identifying the source of these misconceptions, Carlson (1999) noticed that students think of velocity and position graphs as a picture of a physical situation, the path of motion, rather than a set of input and output values that when plotted in XY coordinates generate the graphs. A similar conclusion was reached by Perez-Goytia et al. (2010 October) who contended that understanding the physical concepts is not sufficient for satisfactory interpretation of graphs and suggested putting more emphasis on interpreting graphical representation in science classes.

While motion can be depicted by various functions, piecewise functions are used very frequently in mathematics and physics. Piecewise functions include several function rules restricted to specific domains. Being composed of multiple algebraic rules makes these functions challenging to sketch and interpret (Chazan and Yerushalmy 2003). Some researchers (Hohensee 2006 November) argued that the problem with sketching piecewise function be rooted in multiple domain segments and suggested introducing discrete piecewise functions prior approaching continuous once. Markovits et al. (1986) found out that involvement of more than one rule in denoting a piecewise function makes some students think that a piecewise notation depicts more than one function.

Function continuity and the idea of limits are central in calculus. High abstract load of the definitions followed by an extensive symbolism used to denote algorithms and a lack of connections to students' prior experiences makes these concepts difficult to understand (Bressoud et al. 2017). A common error in students' interpretation of function limits is the notion that functions do not attain their limits values (Szydlik 2000) which can result from a lack of establishing links between function limits values and their applications in real-life situations. Bezuidenhout (2001) argued that the main reason for difficulties in learning the principle of continuity is a deficiency in understanding the underpinning of the limit concept. In a search for methods of improving this understanding, Maharajh et al. (2008) investigated the effects of mapping the concept of the limit image with the definition of limits. Others (Brijlall and Maharaj 2011) proposed highlighting the cognitive and affective domains while proving or disproving function continuity. Fernández-Plaza et al. (2015) investigated students' interpretation of sided limits and concluded that the formal way of evaluating this type of limits does not correspond with the natural direction of function evaluation which caused students' difficulties with computations of these limits.

While analyzing prior research on sketching piecewise functions, investigating function limits and continuity, the idea of applying these principles in the context, for example, motion, was not found. It is hypothesized that embedding the idea of continuity in a purposeful task of construction motion functions will help students realize the importance of the principle and possibly improve the conceptual understanding.

8.3 Formulating a Contextual Framework

This section is to help the instructor guide students through the process of contextualizing the idea of piecewise functions and continuity. It also discusses some methods of introducing the STEM context to math curriculum. This paragraph can also be considered as a reference or a theoretical introduction that is to enrich instructional support during the lab conduct.

8.3.1 Review of Conditions for Continuity Using Function Symbolic Form

Motion problems provide an excellent opportunity for developing the idea of continuity because, for example, a position function must not only be defined at each time instant on its domain but also it must be continuous. These conditions reflect the fact that an object cannot disappear at one location and be recreated at another. The object's continuum of being at specific locations at each time instant during given time interval is supported by science. The precision of graphing piecewise functions when depicting motion requires then several conditions to be met: (a) verification of being defined at each time instant on the domain, (b) adherence to the principle of possessing one dependent variable for each independent, and (c) satisfying a principle of continuity of the graph. All the conditions call for specific procedures to be applied, for example, determining if there are no gaps on the domain of the function or testing if the right- and left-sided limits are equal everywhere, especially at the time of instants where the rule of movement changes. A brief review of these procedures and their interdisciplinary links to physics as follows.

While not all functions must be continuous or differentiable to represent real events, these that are characterized as representing motion must satisfy these conditions. These conditions were intuitively formulated by Leibnitz and called the Law of Continuity (Child 1920). The law of continuity informs that *nature must behave continuously*, which translates to algebra as a condition that functions depicting motion cannot allow discontinuity. In the calculus textbooks, the law of continuity is a part of functional analysis, and it comprises of three different conditions. These conditions merely state that a function must be defined on each element of its domain and must be continuous and differentiable. Should these conditions be met, the law of continuity is satisfied, and the function represents real motion. Following are the details of mathematical symbolism that constitute the first two conditions of the principle. The third condition, the idea of function differentiability, will not be discussed as it would exceed the scope of the STEM lab. It is interesting that function continuity is neither discussed nor emphasized in physics courses and unfortunately graphs presented in physics textbooks often do not satisfy conditions for continuity (Sokolowski 2018). Thus, bringing the concept to students' attention in calculus will benefit their multidisciplinary education. Situating function continuity in a purposely defined task will be used as a tool to guarantee that a formulated algebraic function, $x(t)$, will represent motion according to Leibniz's law. The conditions for function continuity at a given point as they will be used in this STEM activity are defined as follows:

Condition 1: The function must have a value at each time instant, $t = a$, on its domain, including the end of the interval, which can be expressed as:

$$x(a) = \text{object's position expressed in meters, centimeters, etc.} \tag{8.1}$$

Condition 2: The function values cannot have jumped. Thus, the function must have a limit at each time instant on its domain which means that the object must be in every location on the way it is moving. Symbolically, this condition can be expressed as:

$$\lim_{t\to a^-} x(a) = \lim_{t\to a^+} x(a) = \lim_{t\to a} x(a) \tag{8.2}$$

Condition 3: The function limits at each point must be equal to function values at that points which means that a position function must represent a continuous curve:

$$\lim_{t\to a} x(a) = x(a) \tag{8.3}$$

If all the conditions are met, $x(a)$ is classified as a continuous function, thus representing real motion. All these conditions apply to position, velocity, and acceleration functions.

8.3.2 Using the Conditions for Continuity to Formulate Piecewise Position Function

Motion-related problems are often embedded in mathematics curricula as a means of conceptualizing the idea of rates. Because such graphs depict real phenomena, it makes them applicable to exercise more advance mathematics concepts. During this lab, the students will not only use function continuity to construct motion graphs, but they will also translate physical representations of motion into symbolic representations using the idea of algebraic function. Research shows that students' multidisciplinary experiences are more robust if a review of the context that is to be modeled is reviewed prior the activity. Thereby, a discussion about the identifying equivalent meanings between physical and mathematical representations is needed.

In physics, object's position can be expressed using positive or negative values depending on the chosen frame of reference. Position to the left, west, down, or south is usually considered negative, and respectively position to the right, east, north, or forward is considered positive. If the object moves along a horizontal line starting the motion from -4 West, then its initial position is -4 m or just -4. Throughout the lab, students will also use the term velocity which can also be considered positive or negative depending on objects' *direction* of motion. For example, motion to the left will be represented by a negative sign of velocity; -2 m/s and motion to the right will be represented by a positive sign of velocity. Respectively, velocity of -2 m/s represents motion to the left whereas 5 m/s would denote motion to the right if the object moves along a horizontal path. The sign of velocity can also be concluded by the sign of object's displacement.

While the core idea of the lab was modeling motion with various rules of movements, sstudents need to be familiar with the processes of converting kinematics quantities to terms describing functions (linear in this lab) and how to construct an algebraic function for given motion parameters. Table 8.1 shows more details on how to correlate similar meanings. It is suggested that the instructor discusses or even generates such table together with the students and uses its layout while working on constructing specific functions during the lab. The subsequent figures are snapshots of the simulated motion. The teacher will display the simulated motion while the students work in the lab.

Table 8.1 Equivalent meanings between physical and algebraic terms as applied to motion with a constant velocity

Kinematics	Algebraic interpretation	Algebraic representation
Object's position	Dependent variable	x
Position formula	Position function	$x(t)$
Time	Independent variable	t
Constant velocity	Constant slope	m
The initial position Position at any time instant	The vertical intercept of the function A point with given coordinates	b or $x(0)$ (t, x)

Fig. 8.1 Reference line and the man located at the initial position. Source http://phet.colorado.edu

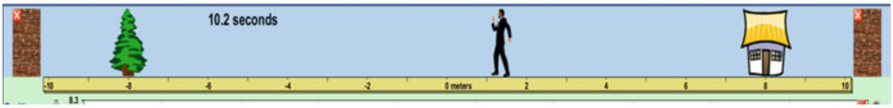


Fig. 8.2 Reference line and the man located at the position of 2 m. Source http://phet.colorado.edu

Following is an example of how to construct a piecewise function for a motion with different rules using the principles of continuity.

Example 1: Suppose that a man starts to walk from a position of 8 m, *left* and he walks for 10 s at 1 m/s in the direction to the right. Then, he turns around and walks at 2 m/s for another 4 s. Construct a position function representing the motion (Fig. 8.1).

Solution: There are two different rules applied during the motion. Therefore, a piecewise function with two different segments will be constructed. It is suggested that the students are provided with a simplified version of Table 8.1 to record given quantities and their values. For simplicity, the table contains only two columns: kinematics quantities and their corresponding algebraic representations.

8.3.2.1 Analysis of the First Segment of Motion

Kinematics	Algebraic interpretation
Initial time instant of the motion	$t = 0\,\text{s}$
Initial position	$x = -8\,\text{m}$
Direction of motion Constant velocity	Positive Slope = 1
Time interval for first segment motion	$0\,\text{s} \le t \le 10\,\text{s}$

The function is defined on its first segment $0 \le t \le 10\,\text{s}$.

Using $x(t) = m(t - t_1) + x_1$, one learns that $x(t) = 1t - 8$ that is valid for $0 \le t \le 10\,\text{s}$.

8.3.2.2 Analysis of the Second Segment (Fig. 8.2)

By reading the given quantities, one can create the following table of symbolic representation for the second segment.

Kinematics	Algebraic interpretation
Initial time instant of motion	$t = 10\,\text{s}$
Initial position	$x = 2\,\text{m}$
Constant velocity	slope $= -2$
Time interval for the motion	$10\,\text{s} \le t \le 14\,\text{s}$

This segment requires a more detailed analysis, and it does refer to the idea of using function continuity. The function must be continuous because it represents a real motion. Therefore, the left- and right-sided limits that represent the man's position at $t = 10\,\text{s}$ must be equal. The left-sided limit, thus the man's position for the time very close to 10 s is $\lim_{t\to 10^-}(t-18) = -8$ m; therefore, the man must be at the same position at $t = 10\,\text{s}$ and that implies that the function must take the same value when the time passes 10 s. Using the idea of right-sided limit, one concludes that $\lim_{t\to 10^+} x(t) = -8\,\text{m}$. This is a very important step that applies the principle of function continuity to find the algebraic representation. Using the coordinate of $(10, -8)$ and $x(t) = m(t - t_1) + x_1$, one learns that $x(t) = 2t - 18$ that is valid for $10\,\text{s} < t \le 14\,\text{s}$.

Using a piecewise notation, the function takes the following form:

$$x(t) = \begin{cases} 1t - 8, & 0\,\text{s} \le t \le 10\,\text{s} \\ 2t - 18, & 10\,\text{s} < t \le 14\,\text{s} \end{cases}$$

8.3.2.3 Verification

The algebraic form of the function can be verified by checking its continuity at $t = 10\,\text{s}$ that can be accomplished by using the three conditions discussed in the Sect. 8.2.1. An example of verifying these conditions is asfollows:

Condition 1: Is the function defined in $t = 10\,\text{s}$? ; $x(10) = 2$; thus, the condition is satisfied.

Condition 2: Does the function have a gap in its range at $t = 10\,\text{s}$?.

$\lim_{t\to 10}(t-8) = 2\,\text{m}$ and $\lim_{t\to 10^+} 2t - 18 = 2\,\text{m}$, since $\lim_{t\to 10^-}(t-8) =$ \$\$ $\lim_{t\to 10^+}(-2t+18) = \lim_{t\to 10} x(10) = 2\,\text{m}$; thus the function does not have a gap thus this condition is also satisfied.

Condition 3: Are the function limits equal to the function value at $t = 10$ s ?

Since $x(10) = 2\,\text{m}$ and $\lim_{t\to 10} x(t) = 2\,\text{m}$; thus, this condition is also satisfied.

Since all the conditions are satisfied, the function is continuous $t = 10\,\text{s}$. This example should serve students as a reference for applying function continuity to model motion using piecewise functions and eventually to become a reference to check any function continuity at a specific point. Students will have a chance to observe a construction of the function graph watching the simulated motion and observe its continuity that is demonstrated in the paragraph that follows.

8.4 Organizing the Lab

8.4.1 Getting Familiar with the Simulation

In addition to exploring function continuity, the simulation helps students to distinguish between the path of motion and the graphs of position (or velocity or acceleration). While the man walks along the horizontal path, his position in XY coordinate system is labeled on the vertical axis and the time of walking on the horizontal axis. Observing the graph production simultaneously observing the man's movement helps to differentiate between position-time graph and object's path of motion and consequently eliminate the confusion between these two fundamental representations.

In the tasks below, students translate between kinematics and algebraic representations and construct position functions for the motion. To enhance the transition to piecewise functions, the students will also be required to state the domain of each function.

The simulation can be played on the classroom screen or students can work independently using the computer lab. It is suggested that the teacher demonstrates at least one example of generating the graph using the simulation (see Fig. 8.3).

After loading the webpage, http://phet.colorado.edu/en/simulation/moving-man, the teacher can select *Charts* from the upper left corner, and the *Introductory* mode which can be used to familiarize students with the simulation controls.

In the simulation, the man's specific position and velocity can be assigned by using the vertical sliders located on the left side. This assigning can also be accomplished by typing the numerical values directly into the highlighted boxes on the left side of the simulation. For a demo, I will use the example from Sect. 8.3.2.

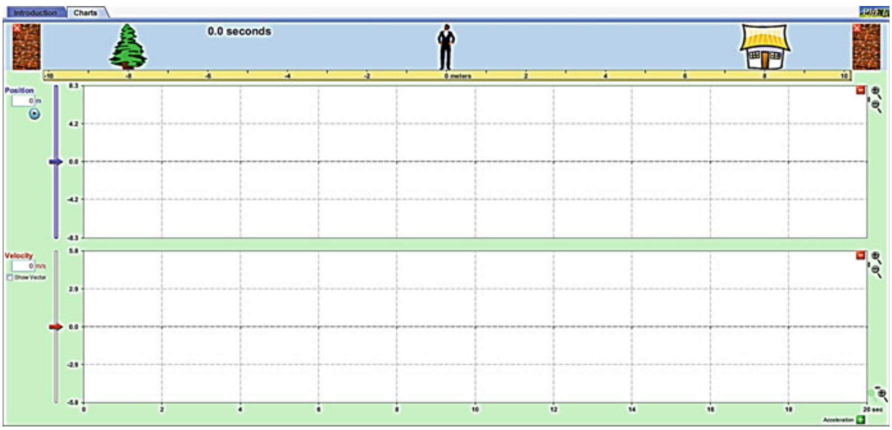


Fig. 8.3 The start-up screen of the simulation shows a man halfway between a tree and a house. Source http://phet.colorado.edu

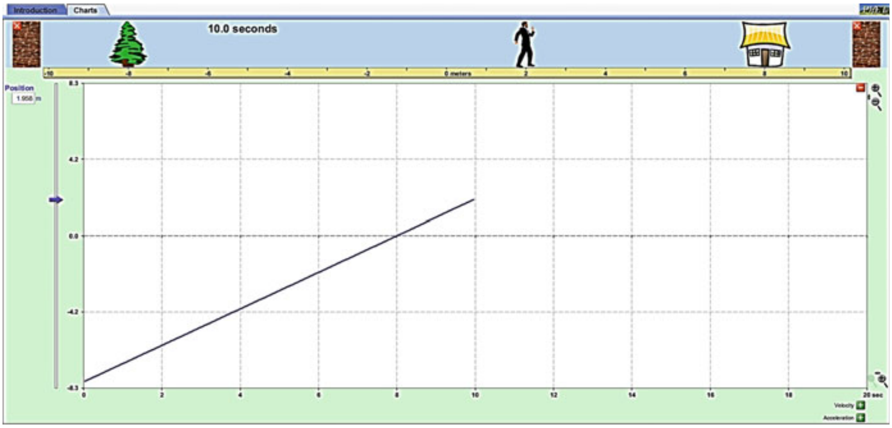


Fig. 8.4 First segment of motion. Source: PhET Interactive Simulations (n.d.)

Example: The man starts to walk from 8 m, *left*. He walks at a speed of 1 m/s in the direction to the right for 10 s. Then, he turns around and walks at 2 m/s for another 4 s. Generate a position-time graph using the simulation.

If the man's assigned position is 8 m to the left, type 8 in the position box and right click. If his velocity is 1 m/s, type 1 in the velocity box and right click. The velocity graph can be hidden by clicking on the radio button showing a red negative sign on the right side of the velocity grid. The man will move after clicking on the *play* button. The motion condition is to be played for 10 s, and this segment of motion is illustrated in Fig. 8.4.

The simulation must be stopped at $t = 10$ s, and the function rule must change at that time instant. This can be accomplished by stopping the man when his time coordinate, as read from the horizontal axis, reaches 10 s. The pause button can be used to stop the motion. Now we enter the next function rule that is the velocity of -2 m/s. Students might be asked questions like *What is the upper limit of the time interval for this segment of motion* or *when should the simulation be stopped*? In this case, attention is drawn to the virtual clock or the final time instant of the motion that is 14 s, and that is illustrated in Fig. 8.5.

8.4.2 Logistics of the Lab

Students will receive a copy of the instructional support and work independently on completing the lab. Before doing so, the teacher opens up the simulation; moving man from http://phet.colorado.edu/en/simulation/moving-man displays it on the classroom screen and demonstrates its features so that the students can use it to confirm their graphs if needed.

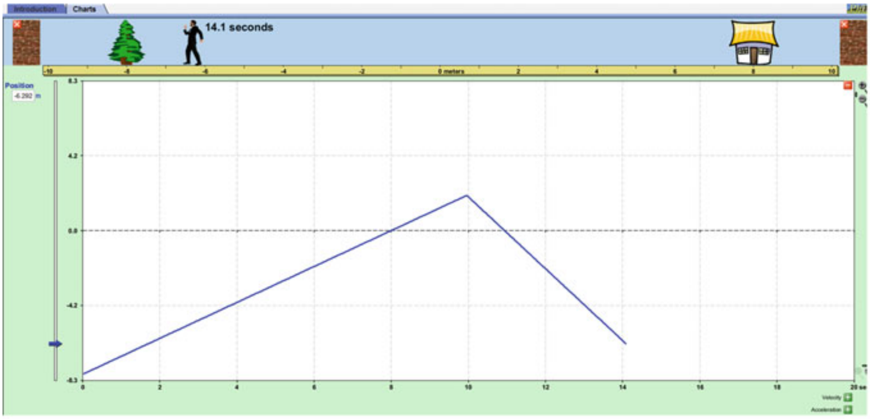


Fig. 8.5 The man turned around and then walked toward the forest. Source: PhET Interactive Simulations (n.d.)

It is suggested that the teacher demonstrates at least one example of generating the graph using the simulation. Sect. 8.4.2 contains all the tasks/questions that the students need to perform or answer. Students might work individually on completing the lab, or they can work in groups. The teacher takes the role of a guide who helps redirecting students' reasoning as needed.

8.5 Lab Outline

Purpose: During this lab, you will construct piecewise functions that will model the motion of a walking man. While constructing the functions, you will apply the idea of function continuity.

Problem 1: There are various types of object's motion: speeding up, slowing down, moving with constant speed and slowing, moving forward and then backward, etc. How do you recognize from the content of the motion description that a piecewise function is to be used to model the motion?

Hypothesis ____

Problem 2: You will observe simulated motion of a walking man and write algebraic equations for his position.

(a) Does the function have to satisfy the vertical line test at any point in its domain? ____

(b) Does the function have to be continuous? What does the condition mean in the context of the motion? ____

(c) Provide a hypothetical motion case when the function does not satisfy this principle?____

(d) Does the function have to be defined at each time instant when the man is in motion? Interpret a case, when the function does not satisfy that principle? ______________________________

(e) Referring to the context of motion, what is the interpretation of the phrase; *position function must have a limit at each point on its domain?* ______________________________

(f) Does the function have to satisfy the principles of continuity? Use the context of motion to support your answer. ______________________________

You will observe several cases of simulated motion. You will transfer the motion parameters into algebraic statements and formulate respective functions symbolically and graphically. All three-function representation, verbal, symbolic, and graphical, must be coherent thus all must reflect on given motion conditions. You will use the condition of the function continuity to support the process of building up the functions. To confirm the functions, you will answer additional questions.

Note:

(a) If a motion is defined on a given interval, it can include time intervals when the object stops. (b) Any motion to the left is denoted by a negative velocity.

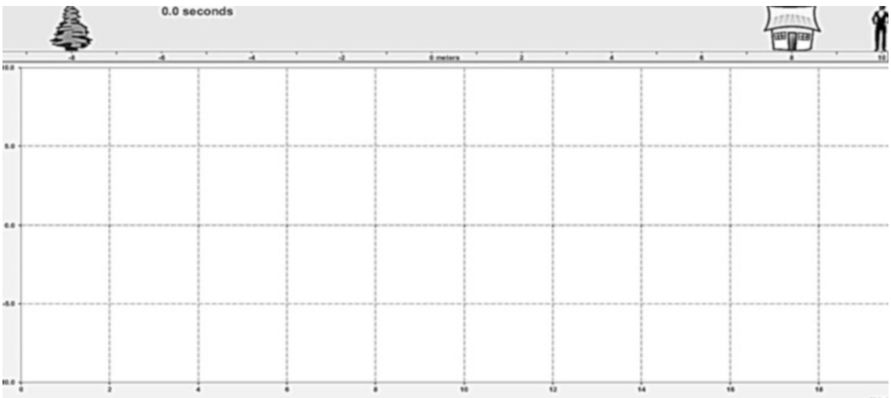
Case 1: The man's initial position is 10 m, left to the reference point. He is moving with a constant velocity of 1.5 m/s *to right* for 8 s and then he stops for the next 6 s.

(a) Do you anticipate a piecewise function to be used to model the motion? Support your answer. ______________________________

(b) If you answered *yes* to the previous question, how many different rules will the function have? ______________________________

(c) What type of functions, linear, quadratic, etc., do you anticipate applying to construct the function? Support your answer. ______________________________

(d) Make a preliminary sketch of the position function on the grid provided below. Note the horizontal axis represents time, expressed in seconds, and the vertical the position, $x(t)$ expressed in meters. Use a ruler to draw the function segments. Make sure that the graph is accurate as you will use it to verify the correctness of the formulated algebraic function.



(e) Does the function appear to be continuous at $t = 8$ s? Justify your answer. ______________________

(f) Is the function defined at $t = 8$ s? Support your answer. ______________________
(g) Write an algebraic representation of the first segment of the man's motion and specify its domain.
(h) Find the left-sided limit of the function at the eighth second of motion. $\lim_{t \to 8^-} x(t) =$______________
(i) What can you learn about the *man's motion* by knowing the limit value? Be specific. ____

(j) What is the expected function value at $t = 8$ s? ______________________
(k) What is the expected right-sided limit of $x(t)$ at $t = 8$ s? $\lim_{t \to 8^+} x(t) =$____
(l) What did you use to support your claims? Choose the answer(s) that apply:

- Function must be defined $t = 8$ s ________
- Function must have a limit at $t = 8$ s _______
- Function must be continuous at $t = 8$ s _______

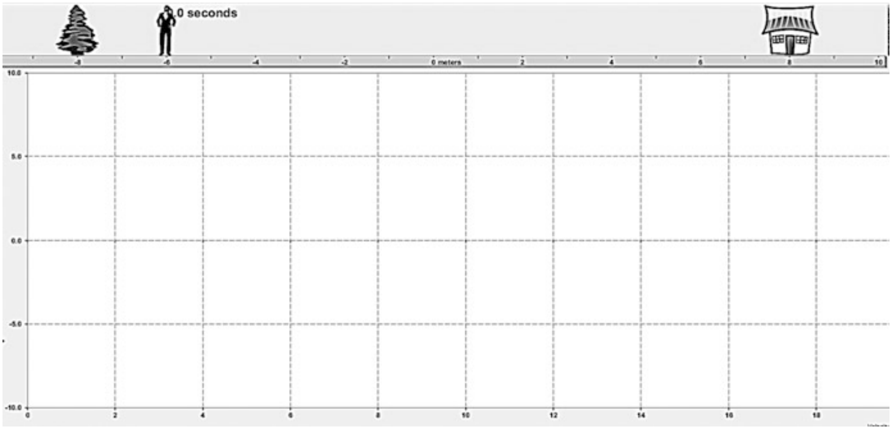
(m) Use the information for (e) and (f) to formulate the initial coordinate for the second piece of the function: ______________________________________
(n) Formulate the second piece of the function and write its specific domain.
(o) Write the entire function using correct piecewise notation. Be specific to the function domains.

To answer the questions that follow, use the piecewise function formulated in (o). Verify if the answers correspond to the function graph.

Calculate the man's position after 4 s of motion.	When will the man reach a position of 5 m to the right of the origin?
Calculate the time when he passes the *origin* (position of 0.0 m).	Prove using a formal procedure that the function is continuous at $t = 8$ s.
What are the domain and the range of the function? Domain: __________range: _________	Suppose that the man stops for 20 s instead of 6 s. Which of the following will change: (a) The domain? _____ (b) The range? ______

Case 2: Suppose that the man is walking at 0.5 m/s for 6 s toward the house starting from the position of 6 m to *the left* of the origin. During the next 4 s, he walks at 1.5 s and finally, he stops for the next 6 s.

(a) How many different function rules do you expect to use to formulate the function? Support your answer. ______________________________
(b) What type of functions, linear, quadratic, etc., will you apply to construct the function? Support your answer. ______________________________.
(c) Make a preliminary sketch of the function on the grid provided below. *Use a ruler* to draw the segments and make sure that the graph is accurate.



(d) While constructing the function, at what time on the man's journey do you anticipate applying the principles of continuity? ______________________

(e) Do you expect the function to be defined at these points? ________________

(f) Construct the first segment of the function and clearly write its domain.

(g) Use the formal statement of continuity to find the initial coordinate for the second segment of the motion.

- $x(6)=$__________
- $\lim_{t\to 6^-} x(t) = x(6) = \lim_{t\to 6^+} x(t) =$__________

(h) Write the initial coordinate for the next segment of motion ______________________

(i) Find the equation for the second segment of the motion.

(j) Write the entire function using a correct piecewise notation. Indicate the correct domains.

These questions pertain to verification of the derived model in (j). Solve the problems and verify the answers with the graph.

Calculate the man's position at 2 s	Find the domain and range of the function
Will the man reach the house located at 8 m? Use the function to prove/disprove your claim	Find the man's displacement between $t = 6\,\mathrm{s}$ and $t = 12\,\mathrm{s}$
Find the time when the man is crossing the reference point $x = 0\,\mathrm{m}$	Prove using a formal procedure that the position function is continuous at $t = 10\,\mathrm{s}$

Refer to the problems from the first page of the lab and confirm or refute your hypothesis.

Problem 1______________________

Problem 2______________________

Provide any suggestions that would improve the learning experience from the lab:

8.6 Posttest Analysis and General Conclusions

There were three free response questions that the students were supposed to answer that targeted their understanding of various aspects of function continuity.
Question 1: *Suppose that you were to construct a position function,* $x(t)$*, for an object moving in the east–west direction. The nature of movement (speeding or slowing or moving with a constant velocity) is not explicitly defined thus any function, trigonometric, exponential, polynomial, or a combination of these functions (piecewise) could be used. The object was in motion on* $t_1 \leq t \leq t_2$*. If an algebraic function is to model the motion, is it sufficient that the function is defined at each time instant on* $t_1 \leq t \leq t_2$*?.*
The answers were clustered into two groups (see Table 8.2) and selected responses from each group were provided, verbatim. Students who answered *yes* were placed in group 1 ($N = 5$, 25%), students who answered *no* were placed in group 2 ($N = 15$, 75%).

This question addressed the first condition for continuity and students' interpretation of this case as it applies to motion. While many of the students claimed that this condition is not sufficient, there were some who thought otherwise. It was interesting to note that these students' verbal supports showed that they possessed a correct intuitive understanding of continuity (see student #2 and #3), yet they associated continuity of function domain with a continuity of function-dependent variable which lead them to incorrect conclusions. These students did not realize that if function $x(t)$ is defined for $t_1 \leq t \leq t_2$ *it* does not guarantee that the function will take all values between $x(t_1)$ and $x(t_2)$ which is the core idea of continuity. Furthermore, these students assumed that *being defined* means continuity of the function range which is not correct. Student 4 claimed that moving object must have a continuous increasing or decreasing graph. While this is correct, motion graphs often include objects' state of rest; thus, this case needs a further reinforcement although the simulation did provide an example of such case.

Table 8.2 Students' reponses to question 1

Student/ Group	Response
1/1	Yes, because the object just cannot vanish
2/1	Yes, $x(t)$ must be defined at all instants $t_1 \leq t \leq t_2$ to reflect reality or else time was skipped and a piece of time cannot be missing
3/1	Yes, because the object is moving in a pattern that is continuous and should be defined at all points (student sketched graphs of continuous increasing and decreasing functions)
4/2	No, it is not sufficient because it could have a graph like this which is not possible (the student sketched a piecewise graph with a jump discontinuity)
5/2	No, the object cannot skip locations
6/2	No, the object position can be defined for all time instants, but the function can have a gap which is not possible (sketched a discontinuous graph)

In sum, these responses also illustrated that the interpretation of the formal mathematical language needs further clarification that will better relate its meaning with students' prior experiences. Although the phrase *function is defined* is often used in mathematics, it seemed that the students were not quite sure how to interpret it when applied to the motion case; did it mean continuity of time or continuity of the man's position or both? Thus, it was further implied that differentiation between *continuity of the function domain* and *continuity of the function range* and merging these two into function continuity needs further elaboration. While typical context-free questions did not exemplify this issue, the STEM context allowed for disclosing these weaknesses.

Question 2: *Does a position function have to satisfy conditions for continuity? Support your answer.*

All students ($N = 20$, 100%) claimed that the motion function must be continuous. However, a more detailed analysis of their supports (see Table 8.3) revealed three different paths of their thinking. Group 1 ($N = 10$, 50%) contained responses (student #1–5) that targeted the idea of position function to be associated with a *continuous* domain. Group 2 ($N = 6$, 30%) contained responses (student #6–8) that intuitively referred to function having continuous y-coordinates, and group 3 ($N = 4$, 20%) contained various answers (student #9–10).

The contexts of the students' responses reflected the principles of continuity; however, it seems that not all the students captured the idea that of all the conditions must be satisfied. Thus, the fact that the object is at a specific location at each time instant does not guarantee continuity of the position graph. Respectively being assigned continuous position does guarantee unique time instants. Associating continuity only with the function outputs, or only with the function inputs is not satisfactory. While the students were more details with their analysis and strived to capture the idea, a room for improvement exists. In sum, the challenge that was not resolved during the lab was how to conceptualize the third condition; $\lim_{t \to a} x(a)$ = $$x(a)$ so that it would resonate with students' prior experiences and make more contextualize sense to them?

Table 8.3 Students' responses to question 2

Student/group	Response
1/1	Yes, because otherwise, it would look like they are teleporting
2/1	Yes, no gaps in time show continuity
3/1	Yes, the man cannot move without time continuing
4/1	Yes, the function must account for all time data
5/1	Yes, the function must be continuous because there are no gaps in time
6/2	Yes, since the object continuously exists, it must have an assigned position
7/2	If the graph never abruptly jumps from one value to another
8/2	Yes, because the object must always be in a position
9/3	The object never disappears, so the function must be continuous
10/3	The time is moving, so the graph must flow smoothly

Table 8.4 Students' responses to question 3

Student	Response
1/1	Sided limits determine if there is a fixed limit by checking if both limits are the same near a point
2/1	Sided limits allow determining if the limits at a point exist. It also provides a good insight how the function acts near a value
3/1	To determine the y-values on both sides of x and to determine if a limit exists
4/1	To understand what the graph does at a certain point depending on which direction you come toward the point
5/1	To write a piecewise function and check the limits
6/2	Sided limit helps us see where a function approaches a hole. We can examine limits even one technically does not exist
7/2	To see which graph satisfies a condition that limit = function value?
8/2	To know how far the function limits go. It helps you get a better understanding of the function
9/2	To know whether the graph is continuous or if it has a hole
10/2	So that you can see any limits that may occur in the functions

Question 3: *What is the purpose of studying sided limits?*
This question was more general, and it was to determine if students related the idea of sided limits with the process of determining function limit. This question did not refer directly to the lab context (Table 8.4).

The responses were also categorized into two different groups. Group 1 ($N = 15$, 75%) that supported their answers using acceptable terminology and contexts (students #1–5) and group 2 ($N = 5$, 25%) that supported their answers using tasks that did not explicitly reflect the purpose of studying and using sided limits (students #6–10). Most of the students from group 1 linked sided limits with verifying function limits at specific points. While the answers were satisfactory and showed that the students did understand the idea of sided limits, surprisingly not many linked the idea with construction of piecewise functions. This shows that there is a need to extend the applications of sided limits in real contexts so that the students consider it as a tool being used in real situations. Thus, more problems or activities should be designed to emphasize the idea that sided limit do support construction of algebraic functions, especially these that have different rules.

This activity showed that contextualizing the idea of sided limits is not an easy task. Even though the students were directed toward using sided limits to construct the piecewise functions, many ($N = 12$, 60%) constructed the position functions assuming it continuity, thus without formal statements supporting the continuity. When situated in context, the idea of function continuity or function being defined took a different more pragmatic meaning and uncovered areas that typically do not surface in context-free problems. For example, response to question 1 showed that the interpretations of the formal mathematical language needs be supported by real-life applications that show how the definitions are applied practically. It seemed that the students were also confused on what the phrase *function is defined* meant.

Differentiation between *continuity* of the function domain and *continuity* of the function range needed more attention to merging these properties in a formal definition of continuity.

Students were engaged in the activity and enjoyed working on it. While this was not examined, an informal discussion showed that observing the simulation helped them to learn how to differentiate between object's path of motion and a corresponding position-time graph. The phase that required more clarification was the labeling of the horizontal axis on the kinematics graphs generated by the simulation.

Students handled questions with piecewise function formulation and sketching better than the ones who were unable to take part in the computer simulation activity. I concluded that this lesson had a positive cognitive effect on students' learning. The element that was especially better handled was considering the end coordinates of a prior piece of the function as initial for the following that students supported by continuity. Most students who did take part in the activity considered incorrectly $y = 0$ as the y-intercept of the following piece of the function.

To provide a better learning experience, I suggest spending an additional instructional unit on practicing writing piecewise function given by graphs and then using the simulated motion to practice continuity. Also, of an advantage can be splitting the activity in two and provide more time for practicing the idea of using a formal notation of continuity which showed a shortfall. After getting acquainted with the formal notation, adding tasks of extracting kinematic quantities from the simulated motion and converting them into algebraic forms is expected to enrich the formal notation. This arrangement is recommended if students have not been exposed to modeling activities before. The next phase of this lab would be to examine how these students handled context-free problems on continuity.

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Chapter 9
Applying Function Transformations to Model Dynamic Systems

Abstract Being able to transform functions due to given conditions is an essential math skill. A preliminary survey of research on teaching transformations has shown that majority of assessment items gravitated toward predicting new graphs due to assigned shifts, compressions, or reflections. Real-life applications of these concepts were rarely discussed. The activity focuses on identifying possible transformations of trajectories of projected objects and constructing new functions. STEM context was provided in the form of a physics simulation *Projectile Motion* that is available for free at http://phet.colorado.edu/sims/projectile-motion/projectile-motion_en.html. Parabolic trajectories were generated by varying the parameters of the object's initial velocity and its relative position. A group ($N = 25$) of pre-calculus students mathematized a *parent*-simulated trajectory and then used it to formulate algebraic functions of other trajectories. It was hypothesized that situating the concept of function transformation in an environment that related to students' prior experience would enhance the purpose of formulating algebraic representations and explicate on applicability of transformation. Analysis of posttest results revealed that situating the learning in realistic contexts brought another dimension to understanding function transformations and infused a deeper understanding of the techniques of constructing transformed functions.

9.1 Prior Research

A preliminary survey of research located using ERIC and Google Scholar has revealed that the studies on functions transformations can be classified into two general categories: as examining the effects of function manifestation using a specific type of transformation or as examining general students' perceptions of the idea of transforming functions given by its symbolic form. For example, Beaudoin and Johnston (2011) used *a purposeful movement* to model shapes of transformed quadratic functions. *The purposeful movement* was directly related to the content being taught, and examples of such movement having students walk around the edge of an object to experience the concept of perimeter. Zazkis, Liljedahl, and Gadowsky (2003) examined students' perceptions of the horizontal

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transformations of the quadratic function, $y = (x - 2)^2$, with a reference to its parent function, $y = x^2$, and concluded that the shift of the parabola is counterintuitive with the students' expectations. They recognized the students' difficulty by contrasting the direction of a horizontal translation with a vertical one, and suggested that there was much more involved in visually processing the transformation of f to $f(x + k)$ than in processing the transformation of f to $f(x) + k$ and suggested creating more studies to examine students' perceptions of function transformations. Baker, Hemenway, and Trigueros (2000) suggested teaching horizontal transformations by using two mental actions: the first that is performed on the function independent variable and the second performed on the function resulting from the first action. Others focused examining how students' learn the idea of stretching or compressing a function. For instance, Sever and Yerushalmy (2007) used technological tools to support the learning of function stretching. They concluded that the technological tools "aroused an on-line sensory stimulus through which they could act in a tangible and concrete way on the abstract functions" (p. 1518). McClaran (2013) pointed out that students often depend on memorized rules for transformations in order to perform the operations and called for including more tasks supporting conceptual understanding of transformations. Hall and Giacin (2013) researched students' tendency to memorize the rules and concluded that when teaching function transformations, instructors usually have students vary the coefficients of equations and examine the resulting changes in the graph which unintentionally leads students to memorizing the rules rather than understanding the algebraic underpinnings. Kimani (2008) argued that while formulating new function students considered these entities as possessing no algebraic relations with the parent functions. It is hypothesized that this case reveals that students do not realize that studying transformations should support and help with new function constructions. Lage and Gaisman (2006) found out that students did not interiorize the effects of transformations on functions when it was needed to think in terms of co-variation between parent functions and transformed functions. Specifically, students had difficulties in identifying what transformation caused a specific change in the parent function and when a transformation was specified they had problems finding its algebraic representation. Nolan and Dixon (2016) called for more research emphasis on how students make sense of the impact of different transformations rather than using procedures that might be poorly understood.

While the studies examined students' perceptions of transformations and the effects the transformations have on parent functions, limited research was located on how students handled constructing algebraic functions using the idea of transformations in real-life applications. Thus, an ultimate question aroused if providing students with real-life application in the form of STEM contexts will enhance the purpose of studying function transformations and support underlying techniques of transforming functions. This study is posited to shed more light on this interface.

9.2 The Lab Objectives and Pedagogical Background

The primary objective examined in this chapter was to learn if students realize that transformations help formulate algebraic representations of real-life situations. Thus, the activity was to support the purpose of studying transformations to efficiently use these properties to formulate algebraic representations of new situations based on being provided a reference function and system's changes. Referring to the physical constraints of the situation, students were also to associate the horizontal shifts with a change of function domain, and a vertical shift with a change of function range. A general screenshot of the lab context is illustrated in Fig. 9.1.

The selection of this simulation is supported by its highly interactive nature that allowed to generate different trajectories of motion, as well as to move the mechanism (cannon) of generating the trajectories (see Fig. 9.4). One aspect that was initially considered an obstacle was a lack of explicitly labeled *XY* axes on the picture. This deficiency was being turned into a meaningful learning experience. In real contexts, *XY* coordinates are typically not shown, and the scientists need to establish one to be able to initiate algebraic analysis of the contexts.

The coordinate system could be established at the position of the cannon (see Fig. 9.1), and this element can also be discussed with the students. All of the didactical advantages of the dynamic system and their effects on supporting the teaching of function graphs were discussed in an earlier publication (see Sokolowski, 2013). This study examines the effects of that system features on students' perception and understanding of transformations.

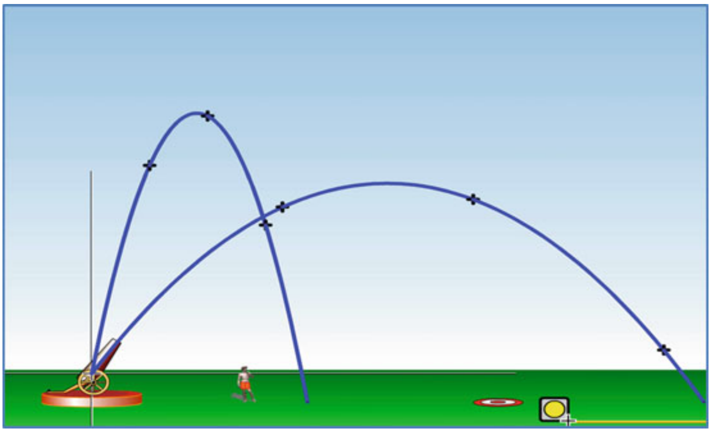


Fig. 9.1 General view of the STEM context. Source: http://phet.colorado.edu

9.3 Merging Mathematical Underpinnings and the Scientific Context

This section provides a background for the teacher on how to merge the context with the underpinnings of function transformations and to guide the students during the activity. The STEM context can be introduced by demonstrating the paths of projected objects depending, for instance, on the initial angle of the projection or the magnitude of the initial speed. Some of the introductory questions can be as follows: can the idea of function transformation be used to construct the trajectories? If so, are there any restrictions on the parameters of the motion to enable this process? Can the motion cases be generated randomly or should the motion be modified with a definite order, patterns, or restrictions? How to decide what general function form to use to formulate transformed functions? By asking such questions, the students are to unpack what they have learned about the properties of quadratic functions and how to relate the properties to physical changes of the observed trajectories. After the discussion, a review of general forms of quadratic functions can follow as suggested herein.

There are specific general forms of quadratic functions: vertex, x-intercepts, and standard. In the vertex form (Eq. 9.1); h and q represent the coordinates of the vertex.

$$f(x) = a(x - h)^2 + q \tag{9.1}$$

In the x-intercept form (Eq. 9.2); x_1 and x_2 represent the horizontal intercepts of the function.

$$f(x) = a(x_1 - x)(x_2 - x) \tag{9.2}$$

In the general form (Eq. 9.3); a, b, and c represent the function parameters called also coefficients.

$$f(x) = ax^2 + bx + c \tag{9.3}$$

The paths of the trajectories generated by the simulation provide enough information to use any of these forms. It could be left up to the students to decide what form they prefer to use. However, due to the activity objectives, the vertex form was suggested because the function formulation during the activity was to be supported by embedding transformations. Students could be given an opportunity to use the other forms as well and compare their results. This option is to help them to realize the advantages of the transformed form and also its limitations.

9.3.1 Discussing and Formulating the Parent Function

While in mathematics textbooks, parent quadratic functions are typically defined with a vertex at the origin, thus $f(x) = x^2$, in real word, this might not be always applicable. Indeed, even in this activity, the parent function, initiated by $f(x) = x^2$, will be defined as a parabola concave down with the coordinates of its vertex representing the position of the maximum altitude of the projected object (see Fig. 9.2). This element can be brought to students' attention, and it will serve as a case for being more flexible with applying function transformations.

Using the trajectory (Fig. 9.2) as a parent function is further supported by a scientific illustration of a projectile motion typically found in physics textbooks that is generated for an angle of inclination between 0° and 90° with *the initial arm* of the angle positioned on the positive x-axis. Thus, the cannon that generates the motion is positioned on the left side, and the object is projected to the right side. It is to note that the marks on the graph represent the position of the object after each second of motion. This property of the motion will not be used for the graph construction, but it will be used to verify the graph correctness.

With such arrangements, the STEM environment is ready to be mathematized. Suppose that the projected object has an initial speed of 18 m/s, and the angle of inclination of 75°. How to use this data to formulate necessary algebraic attributes and formulate a symbolic representation of the path of motion? First, the units of the independent and dependent variables of the sought function must be established. If students use rulers to measure necessary quantities, both x and $f(x)$ lengths will be scaled in centimeters. If students work in the computer lab, they can use the virtual simulated ruler and express the measurements regarding meters. Establishing a frame of reference at the position of the cannon, the horizontal and vertical position

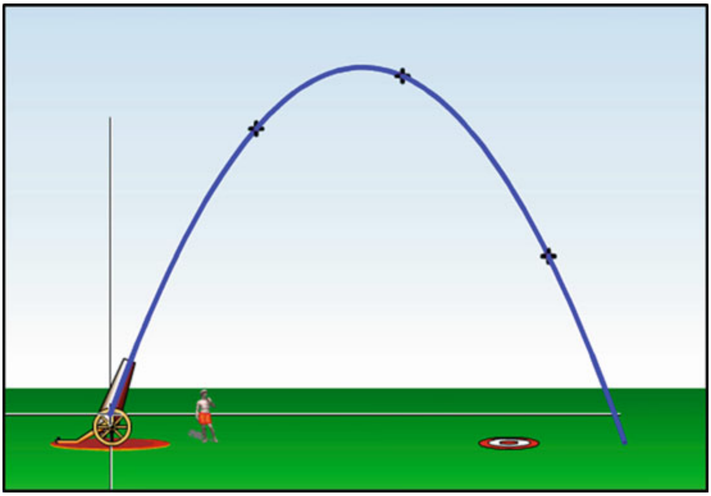


Fig. 9.2 An Image of the parent function. Source: http://phet.colorado.edu

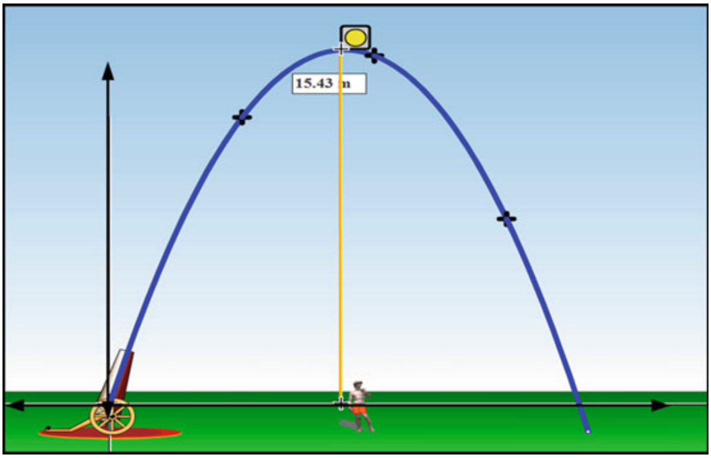


Fig. 9.3 Measuring the y-coordinate of the vertex. Source: http://phet.colorado.edu

of the vertex can be found by direct measuring. The instructor can use the virtual ruler, embedded in the simulation, to demonstrate the process of measuring the necessary quantities. This can benefit some students who might, by mistake, think that measuring the distance toward the horizontal axis will depict the x-coordinate of the vertex and that measuring the distance toward the vertical axis depicts y-coordinate. A teacher's demo of how to take these measurements will eliminate these mistakes.

The process of measuring the maximum altitude, thus y-coordinate of the vertex is illustrated in Fig. 9.3. It returns the value of 15.43 m which when converting to the vertex notation takes the form of $q = 15.43$ m. Since the air resitence is ignored, the parabola is symmetrical with respect to the x-coordinate of the vertex, thus measuring the horiozntal intercepts of the graph and finding their mean value will compute the x-ccordinate of the vertex that is $x = h = \frac{0+16.61\text{ m}}{2} = 8.26$ m. By inserting the coordinates into the vertex form (see Eq. 9.2), the graph of the quadratic function shows as $f(x) = a(x - 8.26)^2 + 15.43$. How to find the parameter a? Students with a physics background might associate the parameter value with the object's acceleration which is not correct because the function does not show a position-time graph but the motion trajectory. Some students will consider evaluating the tangent ratio for the given angle of inclination. This idea is also not correct because the tangent ratio will represent the magnitude of the initial velocity of the object which is not visible in the algebraic model of the trajectory. One of the ways to compute the value of the leading coefficient is to use an additional point from the trajectory and solve the resulting equation for a. Selecting the initial coordinate of (0,0) as representing an additional point and substituting these values into obtained earlier equation produces $0 = a(0 - 8.26)^2 + 15.43$. Solving the equation for a results in $a = -0.23$. Thus, a complete form of the mathematical representation of the trajectory is

$$f(x) = -0.23(x - 8.26)^2 + 15.43 \tag{9.4}$$

How to verify the adherence of the model to the trajectory? The verification is necessary as it will provide means to confirm or refute the model according to the formulated scheme of merging multidisciplinary concepts in STEM developed in Chap. 6. Students can be asked to come up with possible means of the model verification. Most frequently, they would suggest using a graphing calculator and see if the graph resembles the observable path. If a graphing technology is not available, other means must be employed. In this case, students can use, for example, a selected x-coordinate from the graph, calculate corresponding y-coordinate using derived model and then compare it with a respective measured value of the coordinate.

The model (Eq. 9.4) will represent *the parent function* for further function formulation due to transformations. As earlier discussed, the equation does not represent a typical form of a *parent function* found in the textbooks. Considering this case, students are to realize that in real life identifying phenomena that can be modeled by idealistic parent functions is not always possible, yet this does not prevent transformation from applying.

9.3.2 Formulating Algebraic Expressions for Transformed Trajectories

While students realize that a quadratic function could be used to model the trajectories, the movement of the cannon that resulted in modifications of the trajectory provided the basis for further explorations. The simulation allowed for several modifications of the trajectory, for example, moving the cannon upwards and forwards, changing the initial speed and inclination of the projected objects as well as investigating the effects of air resistance on the shape of the trajectory. All these modifications could be embraced in transformations. I suggest exploring one modification at a time and using more complex scenarios when students get comfortable with fundamental transformations. By establishing such sequence, the risk of overloading students' short-term memories will be eliminated. How to categorize the transformations referring to physical changes in the environment? Depending on the mode of the modifications, two general categories of investigations are suggested: (a) investigating the effects of the change of the position of the cannon (b) investigating the effects of change of the parameters of motion, the initial speed of the object and its inclination. The following example discusses the formulation of a new trajectory based on the change of the vertical and horizontal position of the cannon (Fig. 9.4). The original trajectory, the parent function, was left intact to have students refer to it while formulating a new function. By this, students had a chance to compare the properties of both functions from transformation points of view and better identify necessary information. The cannon has been moved to the right and

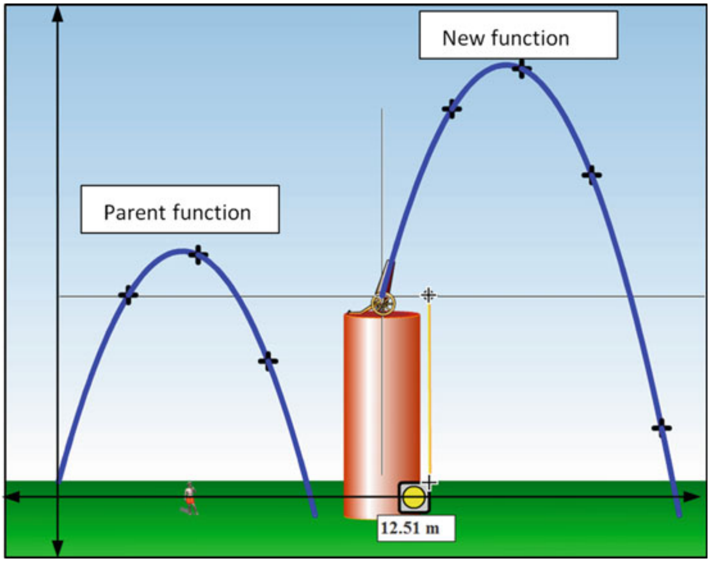


Fig. 9.4 Demonstrating horizontal and vertical shifts of the trajectory. Source: http://phet.colorado.edu

upwards. This movement was activated by left clicking on the cannon and dragging it in the desired directions.

It is to note that the original perpendicular lines in the simulation will move along with the cannon. However, the new function will be built concerning the original position of the cannon and initially made up *XY* coordinates. It is to emphasize that the new function is formulated concerning the originally established coordinates. This element further supports the importance of the frame of reference (coordinates) in formulating new functions using the idea of transformations that gains its meaning in real applications. The teacher might demonstrate—using an embedded ruler—the process of measuring the horizontal and vertical transformation which resulted in $q = 12.51$ m and $p = 21.52$ m and these values constituted, respectively, the vertical and horizontal transformations of the parent function. By referring to Eq. 9.4 and inserting the transformations, the function representing the new trajectory is $f(x) = -0.23(x - 20.77)^2 + 36.96$. Its correctness can be verified in a similar manner as that of the parent function.

Attention can be given to the interpretation of the coefficient of the leading term of quadratic function. The students will explore its behavior in conjunction with function reflections and shifts. One of the objectives of these explorations will be to realize that the leading coefficient does not change even though the position of the vertex changes. The idea was brought up to students' attention and explored throughout the lab's three cases: A, B, and C (see Sec. 9.4.2ts.).

9.3.3 Lab Logistics

When sequenced within mathematics curriculum, the activity can be considered a summary of studying quadratic functions and transformation. The simulation was displayed on a classroom screen. This approach is recommended because it allowed the teacher to demonstrate the context and highlight its necessary features. Alternatively, if preferred, students could work on the activity in a computer lab. As an instrument of data gathering, a metric ruler was used. Students measured the coordinates of selected points using provided screenshots of the trajectories and embraced these measurements in formulating transformations. Before immersing in the activity, students were given an opportunity to observe several simulated motion cases and predict behaviors of the transformed representations by establishing a link between algebraic properties of a quadratic function and the object's path of motion. Once the hypotheses were formulated, students observed the motion, took measurements, mapped the physical properties of the trajectories with transformations and constructed functions that depicted the new trajectories.

9.4 Lab Outline

Purpose In this lab, you will apply functions transformation to model trajectories of a projected object. You will be asked to hypothesize answers to problems related to general properties of the transformed functions and use suggested means to verify your graphs. Before embracing the function constructions, answer the questions that follow.

Problem 1 What function (exponential, linear, quadratic, etc.) can be used to model a path of a projected object? What attribute of the projectile motion did you consider to select the function?

Hypothesis____

Problem 2 Do transformations affect the new function domain and range? If yes, in what respect?

Prediction____

Problem 3 The cannon will be moved horizontally and vertically as well as there will be variations on the initial speed and the inclinations of the projected object to the x-axis. Referring to the chart provided below, identify possible correlated transformations with the indicated motion parameters and write your predictions in the chart.

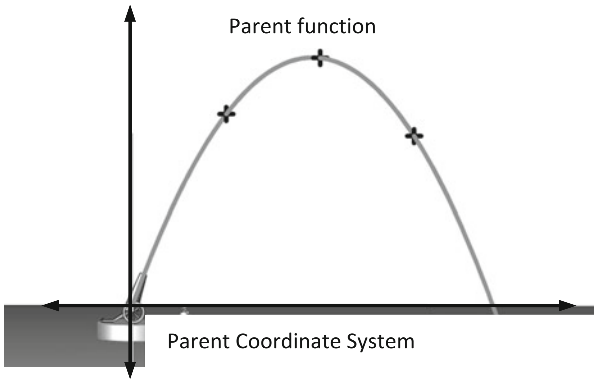
Physical change of the system	Anticipated algebraic transformation
Horizontal movement of the catapult to the right	
Vertical movement of the catapult upward	
Increase of the horizontal component of the velocity	
Changing the angle of projection for its opposite	

9.4.1 Finding Equation of the Parent Function

Take necessary measurements and find an equation of the trajectory of the projected object. Once you have your function formulated, measure the height and width of the picture to establish the window/frame of the calculator so that its entire image shows on the calculator screen. Refer to the labeled *XY* coordinates as a reference for taking the measurements.

Note that the marks on the graph show the position of the projected object after each second of motion. This property will be used later to verify the correctness of the graphs.

Write the dimensions of the window of the graphing calculator: $x_{\min}$=__________, $x_{\max}$=________, $y_{\min}$=____________, $y_{\max}$=____________________



Equation of the parent function:____

- Enter the function into TI (Use *a square* window).
- Write the domain and range of the function considering constraints of the motion.
- Domain:____________________Range:________________________________
- Enter the function in the graphing calculator.
- Did the derived function reflect the image of the parent function? ____If not, correct the function.

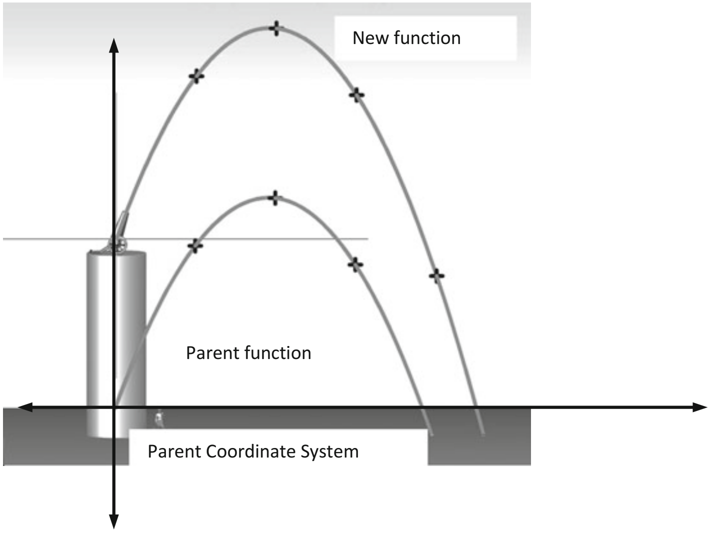
In the examples that follow, the cannon or the parameters of projected motion will change, and you will construct new function equations using the derived parent function. You will also use the derived parent function as a reference function stored in the graphing calculator to investigate the correctness of new functions. Do not delete the parent function from the graphing calculator.

9.4.2 Applying Transformations

Case A: The altitude of the cannon is increased.

The support of the cannon is now vertically extended, however the initial parameters of the motion, thus the initial speed and its direction remain unchanged. For a transformation to be applied, the frame of reference must be left intact, thus it remains on *the ground* (attached to the parent function). The following snapshot illustrates the original path and its transformed position. Do you expect that the leading coefficient of the new function, a, will change? Support your answer. ____

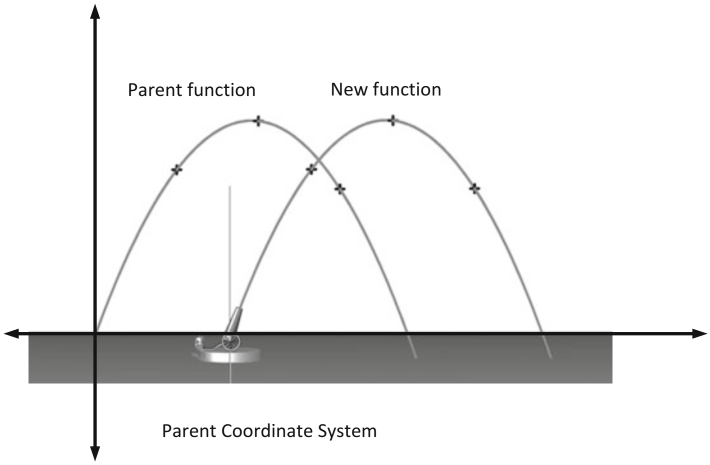
____



- Determine the type of transformation(s) of the new functions that resulted from changing the position of the cannon.________________________________
- Take necessary measurement(s) and using the idea of transformations, construct a function that would reassemble the new trajectory.
- You will now verify the new function. Measure the horizontal and vertical position of the projected object after 1 s of motion (shown by the asterisks) using its path of motion.____
- Use the derived function and *compute* the vertical position of the projected object by entering the horizontal position after 1 s of motion evaluating the function. ____
- Does the computed position correspond with the measured one? ____

- Did the domain of the new function change? ___What is the domain? ____
- Did the range of the new function change?___What is the new range? ____
- Was your prediction about change of function domain and range correct? ____

Case B: The cannon is moved to the right now.
What transformation can be used to model the new path of motion? ____



- Take necessary measurements and find the function equation of the new trajectory. Use the earlier derived parent function and the idea of transformation.

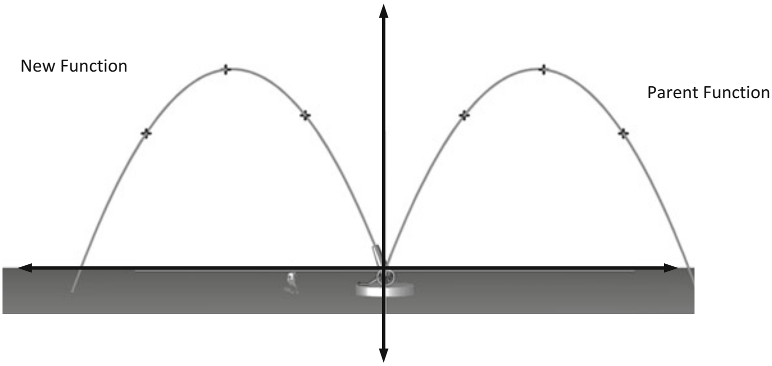
  New function:_____________________________________

- Did the domain and the range of the new graph change with reference to *the parent graph*? What are the domain and range of the new graph? ____
- Did the leading coefficient (parameter *a*) of the new function change? ____
- Did the positions of the *x*-intercepts of the *new function* change? ____
- Verify the graphs by entering *new* function into TI. Make sure that you determine the correct window to observe both graphs; the parent function entered earlier and the new function.
- Do the shapes of the graphs correspond with these sketched on the diagram? ____

- Verify the graph by checking the vertical position (altitude) of the projected object in the new scenario after 1 s of motion using the new function and then by measuring it using a ruler. ___
- Do the heights determined by the two different means correspond? ___

  Comment on any discrepancies ___

Case C: The path is modified by projecting the baseball with an opposite angle, thus by changing the angle of inclination of the cannon (angle of fire) to 110° as measured counterclockwise from the positive side of the x-axis.



Parent Coordinate System

- How is the new graph compared with the parent graph? ___
- What transformation can be used to find the equation of the graph on the left side? ___
- Do you expect a change of the parameter, a, in the transformed graph? ___Support your answer.___
- Determine the equation of the graph of the new function: ___

  ___
- Verify the graphs by entering the new equations into TI. Does the shape of new function reflect the expected one? ___If not, what has to be changed? ___

Additional Questions

1. Suppose that the baseball is fired at 110° in space where there is no gravitational field.

   (a) How will the path of motion of the object look like? Support your answer. ___

(b) Write a function equation for the path of motion of the object. ____

2. Were your predictions to the problems #1–2 correct? ____
____

3. Do your anticipated transformations correspond with the physical changes of the motion (Refer to Chart 1)? Comment if any discrepancies. ____
____

4. Do you have any suggestions on modifying the lab so that you learn more from it?
____
____

9.5 Posttest Analysis and General Conclusions

The students did not encounter significant obstacles while completing the lab. One of the most common questions that surfaced during the lab conduct was what coordinates to use; the one associated with the parent function or the once associated with new function while formulating algebraic equation of the transformed paths. In the simulation, a trace of *XY* coordinates was moved along with the cannon, and this created a doubt. I considered this an opportunity to enhance the notion that new functions, when constructed by using transformations, are due to a frame of reference established while formulating the parent function. Some students noticed that when using coordinates associated with the cannon, the new function appeared precisely in the same form as the earlier formulated parent function. This was especially visible when both functions were viewed on a graphing calculator. The fact that new functions needed to be referred to the same *XY* coordinates as the parent function did not surface when students worked on textbook problems. Thus, the STEM lab helped activate this rule. On the post-lab discussion, the idea surfaced again, and the concept of relative motion or relative position was brought to the students' attention. One of the questions was if formatting a new function due to transformed coordinate system is incorrect? Depending on the purpose of formulating the function, this might be still correct and useful; however, due to practice of the lab objectives, the new function was to be formulated using the coordinate system that were used to formulate the parent function.

While the students correctly hypothesized the applications of quadratic functions to describe the trajectories, only a few ($N = 7$, 28%) supported the function selection by using specific attributes of a parabola, thus attaining an extreme value or possessing a symmetry. Some students used their physics knowledge to justify their answers. Selected responses about justifying the function selection are presented in Table 9.1.

Table 9.1 Students' justifications of using quadratic functions to model path of projected objects

Student	Response
1	The object will go down due to gravity
2	The path looks like a parabola
3	The projected objects follow a quadratic path
4	The speed of the object is decreasing, then increasing
5	The object moves forward and also in the vertical direction that creates the parabola
6	A quadratic function will be used based on the arc of the graph
7	The parabola resembles the path of quadratic function

Luck of using more precise terminology to support the function selection revealed students weak skills of verbally describing the attributes of algebraic functions when a context is used. While all students referred to the graph of a parabola, using the parameters of the real context and explaining the process of mapping them with the properties of quadratic function was not an easy task. A factor that could play a role in formulating the answers was a level and quality of the support that was not explicitly highlighted or taught.

The students predicted that the domain and range of the newly transformed projectiles would change. They did show that both of these critical function attributes can change if the position of the trajectory changes. Most of them stated that the horizontal shift would affect the domain and the vertical range. A few ($N = 8$, 32%) explicitly related vertical transformation or stretch with a change of the function range, and a horizontal compression/stretch with the function domain, respectively. For example "if the initial velocity is higher, the domain and range will be higher" or more specifically "increase of horizontal component of the velocity will increase the function domain." Alternatively, "if the ball is thrown higher it would affect the range, if it was thrown farther, it would affect the domain." Despite the fact that the general properties of the motion were discussed along with the presentation of the simulation, students who were concurrently taking a physics course supported their answers more accurately. The sections that follow provide a more detailed analysis on how the students approached specific cases of transformations.

9.5.1 Analysis of Case A

The students correctly formulated the equation of the transformed graph and majority of them ($N = 15$, 75%) predicted that the leading coefficient of the parabola would not change (see group 2) and the rest of the students ($N = 10$, 25%) claimed otherwise (see group 1). Table 9.2 summarizes students' verbal supports about a possible change of the leading coefficient due to a vertical transformation.

Justifications about the coefficient provided more insight into how students perceive its role in the quadratic function appearance. It was interesting to note

Table 9.2 Students' justifications of the leading coefficient due to a vertical transformation

Student/group	Response
1/1	Yes, because the altitude can affect coefficient
2/1	Yes, because the graphs are different
3/1	Yes, because the parabola got bigger
4/1	Yes, because the other numbers will also change
5/1	Yes, because the parabola is stretched
6/2	No, the vertex is the only thing that changes
7/2	No, because the angle of the trajectory does not change
8/2	No, because the inclination and the initial speed remain the same
9/2	No, the graph is not being stretched
10/2	No, the only position of the vertex changes

that these students who claimed that "a" will change did not refer to the physical properties of the context and its possible influence. The coefficient "a" does not explicitly manifest on the graph unlike the coordinates of the vertex. Discussing its value due to only transformations seems incomplete and does not produce enough prompts for the students to remember the properties. Students who used the STEM contexts were more successful in justifying algebraic properties of the graph. Should such analyses dominate the study of function properties? Examining more examples of similar teaching effects would help in answering such question. Some of the students thought that the new function is wider despite stating that the leading coefficient does not change. In fact, when projected from a higher altitude, the trajectory appeared wider due to the object being longer in motion. This revealed that visual appearance of the new function while displayed in a parent function coordinate system might dilute the function properties. Thus, taking into consideration physical changes of the system are stronger supports for describing embedded transformations.

9.5.2 Analysis of Case B

All students ($N = 25$, 100%) correctly concluded that the function domain will change and that the range will remain unchanged when the function is shifted to the right. They also agreed this time that the leading coefficient of the parabola would not change. While in Case A, all students correctly associated the sign of the transformation with the direction of movement, in Case B some students ($N = 4$, 16%) associated the right shift of the parabola with a negative horizontal transformation although they correctly identified its numerical value and found a correct function representation. This mistake can steam from the counterintuitive the interpretation of f that is to be generated by $f(x - h)$ noted also by Zazkis et al. (2003). If a new function is generated by the $x - h$, then equating this expression to zero and solving for x will provide general means of concluding the correct sign of the transformation, that is $x = h$. Thus, if the transformation is to the right, e.g., 3 units, then $h = 3$, which leads to $x - 3$ and $f(x - 3)$ as an expression for the general form. Verifying the position of the vertex would also help with concluding the final function form.

9.5.3 Analysis of Case C

While students correctly described the physical changes of the graph, reflection about the y-axis, many ($N = 20$, 80%) claimed that the resulting function equation would differ by the sign of the horizontal transformation. More specifically, if $f(x) = -0.58(x - 2.1)^2 + 2.9$ represented the parent function, the students claimed that the reflected function will have the form of $f(x) = -0.58(x + 2.1)^2 + 2.9$. While the function resembled the new graph due to its symmetry, the algebraic form did not follow the properties of reflection about the y-axis that was supposed to be $f(x) = -0.58(-x - 2.1)^2 + 2.9$. Both forms are equivalent and a simple factoring process $f(x) = -0.58[(-1)^2(x + 2.1)]^2 + 2.9 = -0.58(x + 2.1)^2 + 2.9$ proves the equivalency. Some students who attempted to apply reflection did not realize that considering the first operation is replacing x by $-x$ in the new function. Evaluating this case provided prompts for redesigning the structure of this part so that a horizontal transformation cannot suffice the reflection. There was additional question that required the students use their scientific knowledge and construct a function of a path of motion assuming that no external gravitational field was present. Majority of the students predicted that the path will be linear ($N = 17$, 68%); however, the students falsely assumed that the slope of the function will be -0.58 thus equal to leading coefficient of the earlier derived quadratic function. The students did not link the angle of inclination of the path of motion with that tangent ratio representing the slope of the path. The students had not explored the linkage between the angle of inclination and the slope of a respective linear function. This idea was brought up to their attention while the angle inclination and tangent ratio was introduced in the chapter of trigonometry.

This study was undertaken to support the idea that when placed in a STEM environment, function transformations become more tangible and offer opportunities for exploring more deeply their properties and uncover their limitations. Indeed, creating similar labs with other contexts will broaden the idea and have students consider transformations as useful tools to mathematize real phenomena.

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Chapter 10
Investigating Function Extreme Value: Case of Optimization

Abstract Optimization is a central process in engineering designs. Its core idea is rooted in applying mathematics and calculus techniques to finding a maximum or minimum value of a function, often of several variables, subject to a set of constraints. This study investigated how calculus students formulated and analyzed functions that led them to find dimensions of a rectangle that produced a maximum area enclosed by a string of a fixed length. A group of 23 high school calculus students was immersed in an activity that involved hypothesizing possible outputs, direct measurements, data collecting, model formulating, and optimizing the model values. While typical textbook problems on optimization focus students' attention on determining unique dimensions that maximize an enclosed area, this activity extended the exploratory part and underpinned not only the behavior of the function of interest but also the behavior of the constraint functions. This phase helped to disclose potential effects of the constraint functions on absolute maximum or minimum. Posttest analysis revealed that STEM activity not only deepened and helped understanding of underlying optimization processes but also challenged students' mathematical reasoning skills regarded the interpretation of the behavior of the derivative function.

10.1 Prior Research

Optimization is an abstraction of the problem of making the best possible choice from a set of candidate choice (Boyd & Vandenberghe, 2009). It consists of maximizing or minimizing a function value by systematically choosing input values from the set of the allowable domain that corresponds to a maximum or minimum function outputs. Optimization is a field of applied mathematics whose principles and methods are used to solve quantitative problems in disciplines including biology, engineering, physics, and economics. Due to its complexity, optimization is difficult for students. The most common difficulties reported by research are (a) a lack of demonstrating mathematical reasoning skills in understanding the mode of variable change; (b) the inability to use appropriate representation; and (c) overusing procedural approach rather than conceptual (Yang, 2014, July). More specifically,

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Troxell (2002) pointed out certain pitfalls on using technology while teaching optimization and suggested relying more on analysis and validation while concluding a maximum or minimum function value. Poon and Wong (2011) proposed adoption of Polya's problem-solving model to investigate and solve optimization problems in geometry. Schuster (2004) discussed the inclusion of statistical analysis and highlighted the role of combinatorial optimization as a possible area of studies to improve students' optimization skills. Algorithms for optimizing areas enclosed by a fixed length perimeter was investigated by Brijlall and Ndlovu (2013) who concluded that students use "isolated facts and procedures" (p. 16) to optimize problems and linked these shortfalls with traditional ways of teaching these processes. Greenwell (1998) proposed a method that focused students' attention on examining constraints of optimization problems and the geometry before immersing in a formal procedure. Malaspina and Font (2010) examined the role of intuition and rigor in the solving optimization problems and found out that intuition helped formulate the solution process. While analyzing rigor in the solutions, they noticed that students lack justifying skills that would support the optimum solution. Ledesma (2011) investigated the process of identifying a solid with a maximum volume that was built by cutting squares of equal heights from the corners of a piece of cardboard. She concluded that calculus students had difficulties understanding the underlying algebraic algorithm applied to finding a height that produced a maximum volume and suggested using simulations to help students understand the algorithms. Lowther (1999) performed a similar activity with an intent to have students apply a strict algebraic approach. The students, however, chose a method of trial and error. Thus, they missed the opportunity to investigate applications of algebraic functions to find the required volume. The author concluded that "All my students found the volume of their boxes by measuring the other two dimensions and multiplying" (p. 764).

A calculus course encompasses many concepts related to students' prior mathematical background. Thus, the difficulties with optimization might be accounted not only for a high complexity of the optimization problems but also for not providing students with opportunities to merge mathematical ideas in real contexts in prior mathematics courses. The preliminary survey supported the efforts to continue the research on improving the understanding of optimization and students' algebraic techniques. While the STEM context of this study is simple, it will offer students phases of scientific explorations to merge scientific inquiry with mathematical reasoning as it pertains to optimization. With support of the general STEM integrated theoretical framework (Chap. 6, Fig. 6.1), this study was posited to make a transition from observation to mathematization more convincible and more explicit to students.

10.2 Theoretical Framework

This study is an extension of prior research on optimization (Sokolowski, 2015a, 2015b) that was designed for pre-calculus students. While the primary goal of the prior study was to find out if a real context can be used to eliminate students' misconceptions related to optimization, the current one was designed for calculus students and focused on exploring and investigating the behavior of derivatives and their effects on finding the absolute maximum value of the function.

The group of calculus students was not taught optimization problems in their prior mathematics education. Therefore, some elements of the lab activated students' general mathematical reasoning skills by inviting them to formulate hypotheses to the stated problems. A necessary students' calculus background included the techniques of differentiation as well as the principles of the first derivative test to find a maximum function value. Consideration was given to determine whether to place this activity prior a formal introduction to optimization or afterward. While it seemed that either sequence would benefit the learner, this activity was conducted before introducing a formal process of optimization. This arrangement was supposed to reveal the technique of formulating both constraint and the primary functions before delving deeper into the algebraic techniques of optimizing.

Typical problems of optimization found either in algebra or calculus textbooks focus students' attention on finding unique solutions due to formulated algebraic representation that maximizes the scenario (e.g., Stewart, 2007; Sullivan & Sullivan, III 2009). The proposed STEM activity was to extend the exploratory part of the process by including the phase of gathering data, formulating and analyzing a perimeter function, formulating area function based on gathered data, and then finding an optimum solution. Students were invited to hypothesize qualitative and quantitative outputs of the lab and verify their predictions. In the quantitative part, they predicted a numerical output of their investigations, and in the qualitative part, they predicted a general output of their investigations using their reasoning skills. The real context was provided by a string, board, and pins that were used to formulate various rectangles. Students used three different pathways of finding an optimum solution: manual, graphical, and algebraic including use of a graphing calculator.

10.3 Context Development

In addition to having the students explore optimization processes, this activity was to serve as a bridge to similar textbook problems. To allow the link, the following problem, from a calculus textbook (Stewart, 2007, p. 311) was used "Find the dimensions of a rectangle with a perimeter of 100 m whose area is as large as possible," The length of the string was changed to 70 cm due to restricted dimensions of the board. Optimization usually involves formulating more than one function (called constraint functions) that when superimposed generated boundaries on how the function to be optimized behave which in this activity were a perimeter and area functions. Typical support to formulate two (or more than two) function equations is rooted in need to eliminate variables. While from algebra point of view, informing the students about such need can be sufficient, the activity was posited to extend the support to explore not only the nature of the perimeter function but also its rate of change and link these findings with investigating the area function. Thus, on the one hand, the students were to realize that the perimeter is constant for any rectangle, $P = 70$ cm, and the rate of change of perimeter, $P^{'} = 0$. On the other hand, they were to discover that the constant value of the rate can be used to determine how one side of the rectangle changes when simultaneously another does. This phase was to enhance contextual meanings of the variables and their rates. It was also to enhance students' reasoning about the intermediate phases of optimization and highlight the fact that generated rectangles lengths and widths were changing due to a certain pattern that affected the successive phase of finding the area function.

The didactical challenge was how to conceptually link the idea of formulating an expression for the perimeter of the rectangles formulating a function for the area of the rectangles. The link that I have suggested invited the students to establish the width of the rectangle as x, and $h(x)$ as the height and formulating $70 = 2x + 2h(x)$. From there, the students could investigate the behavior of $h(x) = 35 - x$. In this stage, the students merged the idea of perimeter as a formula with a perimeter as a function. Students were to realize a strict dependence of the change of $h(x)$ due to change of x which apparently led to conclude that the maximum area results in a square. The students took the derivative of $h(x)$ and interpreted its meaning in the lab context which occurred not to be that obvious. Taking the derivative of the constraint function is not typically practiced while solving optimization problems; it was seen that this task would not only extend applications of the derivative, but also provide insight about how the variables of interest behave. Students made a preliminary graph of the height function and stated its domain and range. Students then generated a table of values of width versus areas, sketched a corresponding graph (a sample is provided in Fig. 10.1) and found its algebraic equation.

To establish a stronger link between the activity and the problem-solving, the students were suggested to use the x intercept form of the quadratic function, thus $A(x) = a(x - x_1)(x - x_2)$. After substituting the x-intercepts referring to the Fig. 10.1, they obtained $A(x) = a(x)(x - 36)$ whose one factor contained the earlier formulated height function. They used an additional point from the graph and computed the leading coefficient a of the parabola. They used the concept of the derivative again to identify the value of x that produced the maximum area. It was surprising for students to discover that the value of the coefficient was about -1.Due to that the area function took the form of $A(x) = -1(x)(x - 36)$ and then $A(x) = x(36 - x)$ which linked the form with traditional setup of such problems found in calculus textbooks. Students used also a graphing technology to find the best fit curve for the data and then analyzed it. A diversity of routes taken in this lab summarized in Fig. 10.2 were to highlight correlations between all the representations and enhance the algebraic reasoning of these processes.

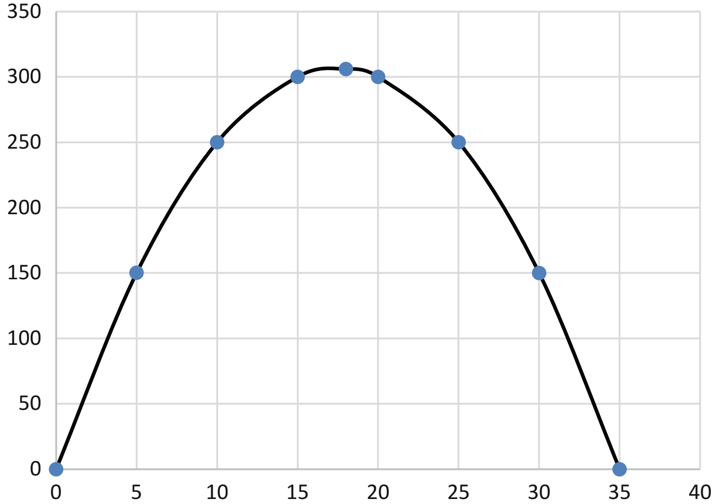


Fig. 10.1 Width versus area graph

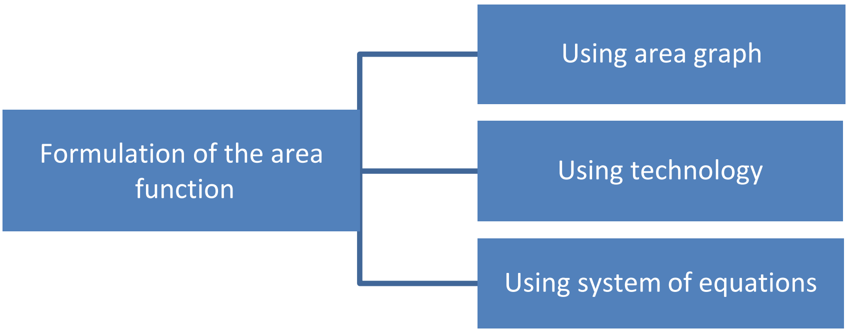


Fig. 10.2 Diversity of representations used to formulate the area function

10.4 Setup, Materials, and Lab Logistics

A science lab was designated for the lab conduct because the students could be seated in groups, which nurtured ideas exchange. This lab could also be conducted in a regular math classroom. There were five groups of four students and one group of three students. The activity lasted for one class period (55 min) that allowed for taking data, generating the graph and initiating the analysis. Students who did not finish the lab in the class took it as homework. It is suggested designating about 90 min in class for the lab completion. Each group was given a string of lengths between 70 cm and 80 cm, a Styrofoam board of 40 cm by 60 cm, four pins, a metric ruler, and a set of French curves to allow sketching the best fit curve.

The instructor initiated the activity by explaining that the students would investigate the area enclosed by a string of a fixed length. The instructor demonstrated various rectangular shapes generated by changing the sides of the rectangles by keeping the string length constant. He explained that the initial shape of the polygon is already prearranged for the student and then he turned the students' attention to the problems listed in the activity and invited them to hypothesize the answers. Although the students took data in groups, each student was responsible for formulating hypotheses and answering the lab questions individually.

10.5 Lab Outline

Objective During this lab, you will explore changes of the dimensions of rectangles with a fixed perimeter and explore the effects of that processes on the area enclosed.

Materials String, board, pins, a ruler, French curves, TI-84.

Problem 1 Suppose that you are given a string of a fixed length. If the string is to form rectangles, how will the height of the rectangles change if the width increases by 1 cm?

Hypothesis ______________________________

Problem 2 What will the derivative of the perimeter function represent? What is the expected algebraic form of the derivative?

Hypothesis ______________________________

Problem 3 Will the area be enclosed by a string of a *fixed length* change if the dimensions of the rectangle change? Support your justification.

Hypothesis ______________________________

Problem 4 Suppose that the area is labeled on the vertical axis and the width on the horizontal what type of graph will be generated if respective data is plotted?

Hypothesis ______________________________

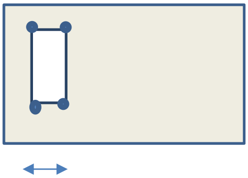
10.5.1 Lab Procedure

You will take the data using the given Styrofoam board, pins, and the string. Note that the first setup to take data is prearranged (see the diagrams below). You need to measure the heights of the resulting figures, compute the areas, if possible, and record these in Table 10.1.

Table 10.1 Generated data for the lab

Perimeter (cm)										
Width (x) of the rectangle (cm)	0	5	10	15	18	20	25	30	35	
Height (h) of the rectangle (cm)										
Area (A) of the rectangle (cm^2)										

- Place another pin at a distance of 5 cm from the existing one (see the next diagram). Use another two pins and wrap up the string around to form a rectangular polygon. Measure the height and find the area.



5cm

- Change the length of the base of the rectangular polygon as indicated in Table 10.1 and repeat the steps until the rectangle reaches the width of 35 cm.
- Add another measurement if the string is longer than 70 cm.
- Record the perimeter of each figure (one more cell is provided for an additional dataset if needed).

10.5.2 Data Analysis

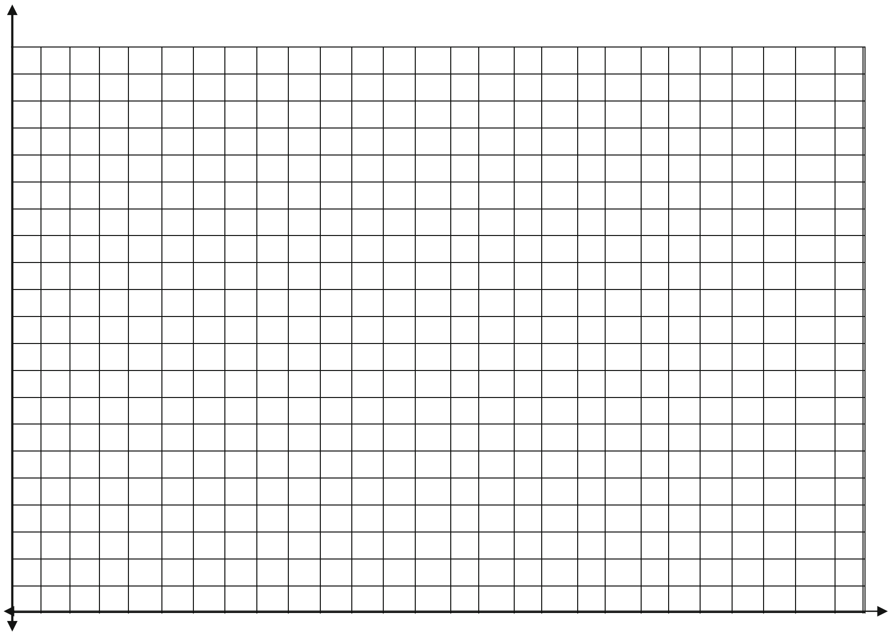
Part 1: Analysis of the Constraint Function

- Was the perimeter of the rectangles constant as you modified the rectangles' dimensions?
- What is the value of the perimeter and its rate of change? ____________________
- The heights of the rectangles were changing by a specific rule depending on the value of x. Using the constant value for the perimeter, 70 cm, x for the width, and $h(x)$ for the height, construct a perimeter function: ________________________________
- Solve the expression for $h(x)$____
- Take the derivative of $h(x)$; $\frac{dh(x)}{dx} =$____
- What is the interpretation of the derivate? ___________________________________
- Make a preliminary sketch of $h(x)$ and identify its domain and range referring to constraints of the lab.

- Were your hypotheses to Problem 1 and 2 correct? ______________________

Part 2: Generating and Analyzing the Area Function

- Is the area constant as the width of the rectangle increases? ________________
- Was your hypothesis to Problem 3 correct? _____________________________
- Are there some specific dimensions that produce a maximum area enclosed? ____
- On the grid provided below, plot *Area* versus *width*. Label the area of the rectangles on the vertical axis and the width on the horizontal axis. Establish a scale so that the graph fills in the entire grid. Use French curves to draw a smooth graph.



- You will find an algebraic equation of the area function using $A(x) = a(x - x_1)(x - x_2)$ where x is a general variable representing the width of the areas and x_1

and x_2 represent the horizontal intercepts of the parabola. Use an additional point from the graph to calculate the value of the parameter a and then formulate the area function.

- $A(x) =$ ____
- Find an expression for the rate of change of the area with respect to the width and sketch it below. What are the units of the derivative? What is physical interpretation of the units?

- Use the first derivative test to determine the value of x that produces a maximum area of the rectangle. ____________________________________
- Do the signs of the derivative correspond with the behavior of the area function as seen through calculus methods? Support your answer. ______________
- Does the computed width correspond with the one hypothesized using the table of values and the graph? ________________________________
- What is the maximum area? ________________________________

10.5.3 Model Verification Using Graphing Technology

During this part, you will generate the function equation using the data from Table 9.1 and a graphing technology. How to enter the function and find its algebraic expression?

- Press STAT, enter the length in the column under L1, and the calculate area under L2.
- Quit, press STAT again, and navigate to CALC (top middle function in the menu).
- Press CALC, examine possible regression curves, and select the one that you anticipate will best follow the generated graph properties.
- Press enter.
- Retrieve the algebraic equation of the curve. ______________________
- Take the derivative of the function. ____________________________
- Does the maximum value of the area correspond with the one computed using the earlier derived area function by hand? ________________________

10.5.4 Merging the Lab Outputs with Techniques for Optimization

In the textbook problems on optimization, constructing area or volume function by taking data and sketching graph is not often suggested to use. Therefore, more general techniques of solving optimization problems were developed. The questions that follow attempt to etablish the link. The value of the parameter a should be close to -1 and thus the area function should be in the form $A(x) = -1(x)(x - 36)$ or $A(x) = (x)(36 - x)$.

- Do you agree that the area function can be constructed using a general formula for area of a rectangle that is $A = (w)(l)$?____________________________________
- If you choose w as a width, what is the expression for l? ____________________
- What are the differences and similarities between $A(x) = (x)h(x)$, $A(x) = (x)(36 - x)$ and $A = (w)(l)$?____

Provide lab reflections and possible suggestions for its improvement:

____
____
____

10.6 General Discussion

All students completed the lab. They followed suggested paths of analysis: the table of values, the graph, and the analysis of the first derivative test and concluded that the maximum area was produced when the width of the rectangle was about 17 cm. The virtue of completing the lab and finding rectangle dimensions that maximized the enclosed area could support the premise that merging systematic scientific methods with algebraic techniques can serve as a learning tool. This lab also helped reveal several factors that pertained to derivatives that are typically not activated while working on textbook problems. These factors were visible when (a) students converted the perimeter formula to a function and were asked for an interpretation of the derivative and (b) when they used the derivative to analyze the area function. Both these factors did not affect the final answer. However, discussing these phases with students after the lab supported the hypothesis that extending the inquiry of the activity and immersing the students in a deeper algebraic reasoning helped to correct certain misconceptions. Consequently, it also solidified students general disposition about their readiness to apply calculus tools in real contexts.

10.6.1 Formulas Versus Functions

One of the factors that the students did not feel comfortable with was the phase of converting a perimeter formula to a function. While they correctly predicted that the rate of change of height decreases while the rate of width increases, many ($N = 10$, 43%) did not recognize that the rate of perimeter function should be zero due to the perimeter of all the rectangles being constant. This finding prompted to contrast the concept of formula and function with the students after the lab. If $P = 2w + 2l$ is given, students typically do not associate this expression with an algebraic function because the purpose of using it is to compute perimeter for unique sides values rather than using it to function analysis. Should calculus students be fluent in recognizing the differences between function and formula and be able to make transitions between these two fundamental algebraic representations? Should there be designated instructional units in mathematics that would explicate on the differences and provide means of transitioning between formulas and functions? The group of students was also asked to contrast formula for an area of a rectangle with the derived function from that data. Their responses lacked details and a clear understanding of the close linkage between these representations which suggest to emphasis more the similarities and differences between formulas and areas in school practice. While the claim can be premature, further research in the domain of formulas versus function not only as it applies in math but also in science seems as a natural consequence of the lab outputs.

10.6.2 Sketching and Interpreting Rate of Change

The other aspect that showed up during the analysis was the interpretation of the rates, sketching the derivative of the area, and using the first derivative test to identify the dimensions of the maximum area. The students used the first derivative test to solve real-life textbook problems prior the optimization lab. Thus, they were familiar with its application. During the lab, many of them ($N = 16$, 70%) did not sketch the derivative of the area on its entire domain but only where the area had a positive rate thus $0 \leq x \leq 18$ cm or $0 \leq x \leq 36$ cm. I implied that they did not feel comfortable to admit that rate of change of a real quantity can be negative. A post-lab discussion had supported this claim. Revealing this shortfall signified high educational values of real STEM contexts. It also illustrated a barrier that perhaps exists between calculus tools and their real applications amplified a need for designing more labs where the students would experience direct applications of calculus in real life. Such limitations did not surface in students' work when analysis of pure functions was concerned and did not surface current research on understanding the concepts of calculus. It seems that interpreting just a negative slope is not sufficient, students need to be given real contexts when negative rates will result from the analysis. Another element that showed up during the lab analysis was associated with handling of the physical units of the derivative. In science, simplifications of units is a common practice. For example, if using $x = vt$ students are required to prove by simplifying the units of velocity and time that the unit of the distance denoted by x is *meter* $[x] = \frac{\mathrm{m}}{\mathrm{s}} \cdot \mathrm{s} = \mathrm{m}$. Yet, it occurred that simplifying the units of the derivative might lead to units that are difficult to interpret in real contexts. For example, the students were to state and interpret the meaning of the derivative of the area function with respect to width. Several of them ($N = 10$, 43%) stated that the unit $\left[\frac{dA}{dx}\right] = \frac{\mathrm{cm}^2}{\mathrm{cm}} = \mathrm{cm}$ and claimed that the unit represents "change in area in terms of width" which is correct however the unit of centimeters is misleading. Retaining the final units as $\left[\frac{dA}{dx}\right] = \frac{\mathrm{cm}^2}{\mathrm{cm}}$ would provide better means for a more realistic interpretation. In a similar fashion, many of the students ($N = 20$, 87%) reduced the units of meters in $\frac{dh(x)}{dx} = -\frac{\mathrm{m}}{\mathrm{m}} = -1$ and were unable to properly interpret the result in that context. These observations prompted me to address these issues and point out that while deriving the units of the derivative, simplification of the units of the component quantities is not recommended, because this process might scrap the rate from its important contextual meaning and lead to units that are difficult to interpret.

10.6.3 Transitioning to Textbook Problems

The STEM lab was also to serve as a reference to solve textbooks problems on optimization. Thus, while introducing other types of problems on optimization, I referred to the activity emphasizing that the function be optimized usually included a constrained function that stemmed from the conditions embedded in the problem. Inducing and maintaining parallelism between the lab and the textbook problems was well received by the students. When quizzed, the students handled very well similar area/perimeter problems. However, they were not equally successful on problems, for example, requiring finding dimensions of a rectangle inscribed under a graph that would result in a maximum area enclosed. This context would need perhaps another real experience to align its technique of solving with students' prior experiences.

10.6.4 General Conclusions

Foremost, it is seen that the students should be given opportunities to explore similar contexts, e.g., areas and perimeter in their earlier math classes, so that when they reach calculus, the exploratory analysis can focus more on using calculus tools. This shift would also allow for conducting more activities in different contexts during the course conduct and exemplify general patterns of the techniques. It is also seen that placing more attention to analyzing the rate of change of the constraint functions would deepen the analysis because it uncovered the relations between their rates of change, thus shed for light into possible dimensions of the quantity of interest. At last, this study also supported the conclusion reached by other researchers (e.g., Ledesma, 2011) that the source of difficulties in solving calculus problems might not necessarily be rooted in the mathematization of the processes, but in the difficulties of understanding the underlying mechanisms of the mathematical principles and apparatus applied.

It seems that STEM contexts display a great potential to help the learners to understand the mechanisms.

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