Using Puzzles, Paradoxes, Provocations and Sophisms for Enhancing Teaching and Learning of Engineering Mathematics

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Australian & New Zealand Industrial & Applied Mathematics

Topics for EMAC2015 include, but are not limited to:

- Biomedical
- Chemical engineering
- Computational fluid dynamics
- Differential equations
- Dynamical systems
- Engineering mathematics
- **Engineering mathematics education**
- Environment
- Financial engineering
- Mathematical biology
- Non-linear systems
- Operations research
- Optimisation
- Stochastic and statistical modelling

http://www.ams.org/msc/

2000 Mathematics Subject Classification (MSC)

97: Mathematics Education is an area newly added to the MSC effective with the 2000 revision, but of a long heritage. Topics of discourse cover all levels of mathematics education from pre-school to university level, and can focus on the student (through educational psychology, for example), the teacher (continuing development or assessment), the classroom (and the books and technology used in the classroom), or the larger system (policy analysis, cross-cultural comparisons, and so on). Analysis can range from small case studies to large statistical surveys; perspectives range from the fairly philosophical to the clinical.

PBRF Guidelines 2012

Mathematical and Information Sciences and Technology

Applied mathematics includes the development of, the analysis of, and the solution or approximate solution of mathematical models including those arising in physical, geophysical, marine and life and health sciences, engineering and technology; it also includes the development and application of mathematical theories and techniques that further these objectives. The Mathematical and Information Sciences and Technology panel will also consider operations research and optimisation, including deterministic and stochastic models and solution methods. This subject area also includes mathematics education.

Developing Digital Resources of Puzzles, Paradoxes, Provocations and Sophisms in STEM Subjects and Investigating their Impact on Students' Engagement and Retention

LTDF Project Team and their Areas

A/Prof. Sergiy Klymchuk (Project Leader) – **maths and engineering**

Prof. Sergei Gulyaev – **physics and astronomy**

A/Prof. Nurul Sarkar, Dr Jacqueline Whalley, Anne Philpott – **computer science**

STEM subjects – ‘too dry and boring’

Emotional disengagement and academic disinterest

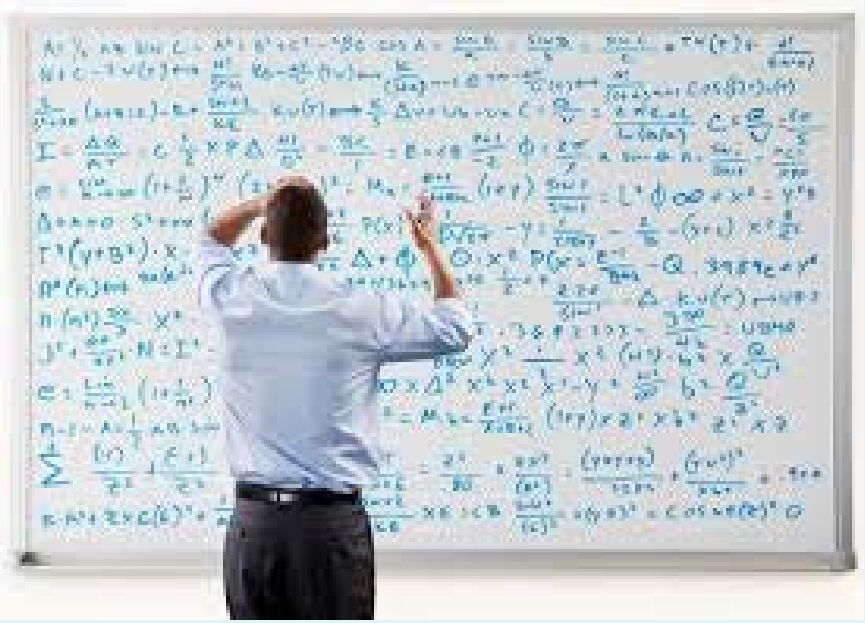
- **Engage** students’ emotions, creativity and curiosity
- **Enhance** critical thinking and problem-solving skills
- **Increase** the retention rate

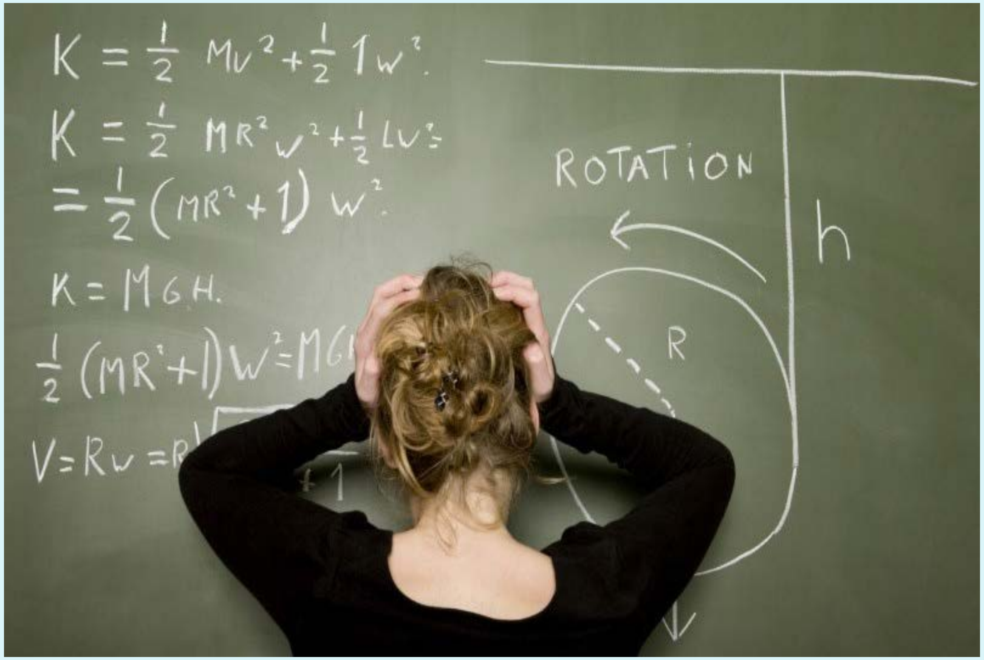
If the students are interested the rest is easy

“A student is not a vessel to be filled, but a fire to be lit”

Plutarch

Perception of Maths





Maths is a Way of Thinking

- Looking for evidence and proof
- Not taking anything for granted
- Logical reasoning
- Conceptual understanding
- Critical/analytical thinking
- Paying attention to details

Useful skills in other areas of life!

Finding a Pattern is not Enough!

$$\lim_{x \to 1} \frac{1}{(x-1)^2} = \mapsto$$

???

$$\lim_{x \to 8} \frac{1}{(x-8)^2} = \infty$$

!!!

Why is Attention to Details Important?

- All conditions of a theorem;
- The locality (interval/point) of a statement;
- The properties of the function involved;
- The shape of the brackets;
- The word order;
- Etc.

Why bother?

Puzzles vs routine problems

A puzzle is a non-routine and not standard problem presented in an entertaining way

Simplicity: Easy to state and remember and looks deceptively simple

Surprise: Teases by a surprising solution and an unexpected answer

Entertainment: Fun to solve

More about puzzles

- Puzzles represent “unstructured” problems
- Puzzles teach how to think “outside the box”
- Puzzles illustrate many general and powerful problem solving principles

Edutainment = Education + Entertainment

Puzzle-Based Learning

Promoted by Z. Michalewicz (2008, 2014)

- Meyer, E.F., Falkner, N., Sooriamurthi, R., Michalewicz, Z. (2014). *Guide to Teaching Puzzle-Based Learning*. Springer.
- Michalewicz, Z. & Michalewicz, M. (2008). *Puzzle-Based Learning: An introduction to critical thinking, mathematics, and problem solving*. Hybrid Publishers.

- **Paradox** - looks invalid but is in fact true
- **Provocation** - looks routine but has a catch
- **Sophism** - looks correct but contains a deliberate flaw

Evaluation

- Resources: e-books for mobile devices and for institutions' online platforms
- Effectiveness - evaluated via: comprehensive questionnaires, interviews, class observations and analysis of course success rates
- Benefit for learners - enhanced engagement, better conceptual understanding
- Expected outcome - increase of the retention rate

More Benefits for Students:

Puzzles at Job Interviews

"Now more than ever, an education that emphasizes general problem solving skills will be important".

Bill Gates

Many companies use puzzles at their job interviews to evaluate candidate's problem solving skills and select best of the best.

Puzzles at Job Interviews

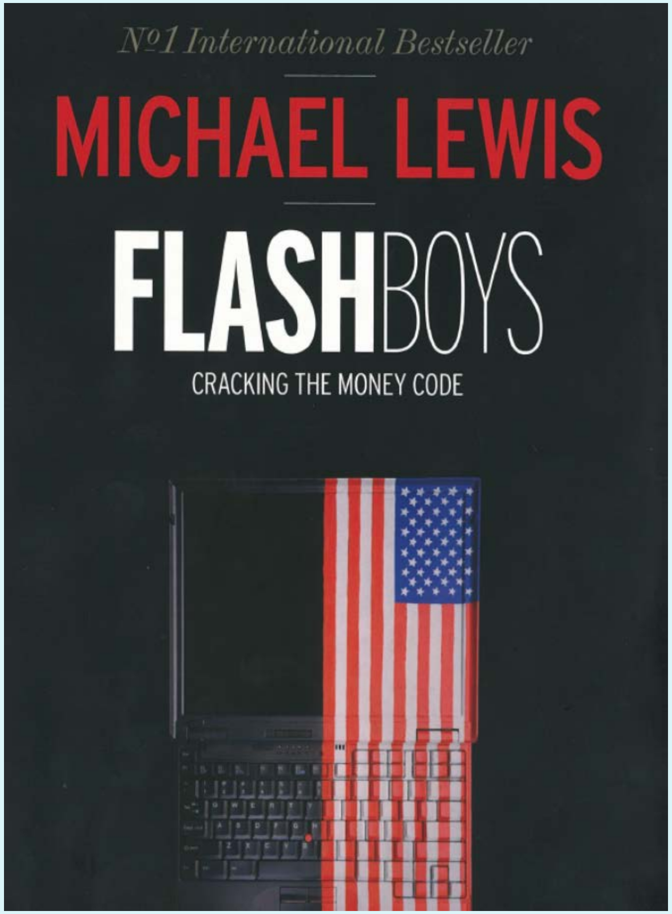
"The goal of Microsoft's interviews is to assess a general problem-solving ability rather than a specific competence. At Microsoft, and now at many other companies, it is believed that there are parallels between the reasoning used to solve puzzles and the thought processes involved in solving real problems of innovation. You have to hire for general problem-solving capacity."

Poundstone, W. (2004). How would you move Mount Fuji?

Real Examples

From the book "Flash Boys" by Michael Lewis, 2014

Hiring a programmer
for a financial company
on Wall Street, New York
with the annual salary
of 270,000 USD



Real Examples: Question 1

Question 1: Is 3599 a prime number?

This is where the boring formula from algebra

$$a^2 - b^2 = (a - b) \times (a + b)$$

can help you to become rich ☺

$$3599 = (3600 - 1) = (60^2 - 1^2) = (60 - 1)\,(60 + 1) = 59 \times 61$$

Real Examples: Question 2

sions. “He says there is a spider on the floor, and he gives me its coordinates. There is also a fly on the ceiling, and he gives me its coordinates as well. Then he asked the question: Calculate the shortest distance the spider can take to reach the fly.” The spider can’t fly or swing; it can only walk on surfaces. The

Real Examples: Question 2

shortest path between two points was a straight line, and so, Serge figured, it was a matter of unfolding the box, turning a three-dimensional object into a two-dimensional surface, then using the Pythagorean theorem to calculate the distances. This took him several minutes to work out; when he was done, Davidovich offered him a job at Goldman Sachs. His starting salary plus bonus came to $270,000.

Paradox – Torricelli's Trumpet

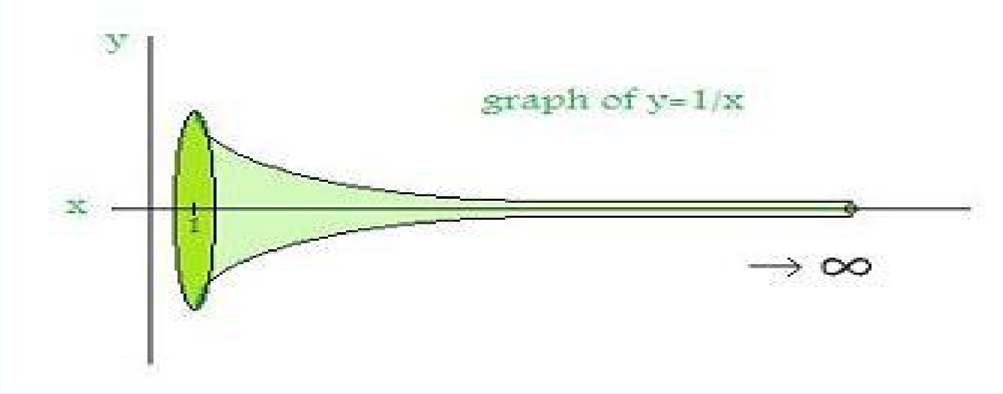
There is not enough paint in the world to paint the area bounded by the curve $y = \frac{1}{x}$, the x-axis, and the line $x = 1$:

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{b\to\infty}(\ln b - \ln 1) = \infty.$$

However, one can rotate the area around the x-axis and the resulting solid of revolution would have a finite volume:

$$\pi\int_1^\infty \frac{1}{x^2}\,dx = -\pi\lim_{b\to\infty}\left(\frac{1}{b} - \frac{1}{1}\right) = \pi.$$

One can fill the solid with π cubic units of paint and thus cover the cross-section area with paint.



Paradox – Torricelli's Trumpet

We should differentiate the "mathematical" universe from the "physical" universe. From a mathematical point of view one "abstract" drop of paint is enough to cover any area, no matter how large. One just needs to make the thickness of the cover very thin. Let's say we have 1 drop of paint of the volume of 1cm^3 and we need to cover a square plate of the size x by x cm. Then the (uniform) thickness of the cover will be $\frac{1}{x^2}$ cm. If x = 100cm then the thickness is $\frac{1}{10000}$ cm. If $x \to \infty$ then the area $x^2 \to \infty$ and the thickness $\frac{1}{x^2} \to 0$. But at any stage the volume is $x^2 \times \frac{1}{x^2} = 1$ cm^3. So mathematically you can cover any infinite area with any finite amount of paint, even with a single drop. In reality such infinite areas don't exist, nor can one make the cover infinitely thin.

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \ldots = 0$$

$$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \ldots = 1$$

So, 0 = 1 !!!

How about that:

$1 - 1 + 1 - 1 + 1 - 1 + 1 - 1 \ldots = S$

$1 - (1 - 1 + 1 - 1 + 1 - 1 + 1 \ldots) = S$

$1 - S = S$

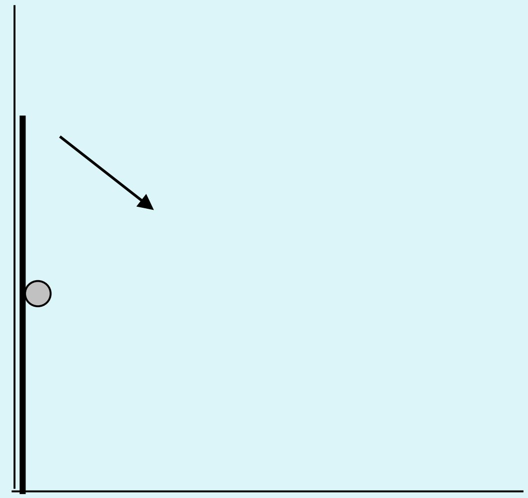
$2S = 1$

$S = ½$

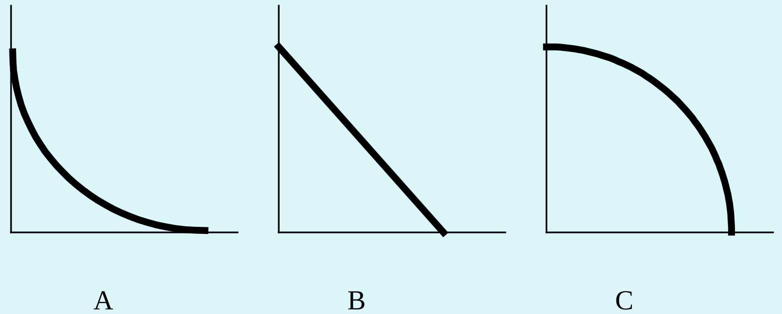
Absurd?

But – Euler and Leibnitz did believe so!!!

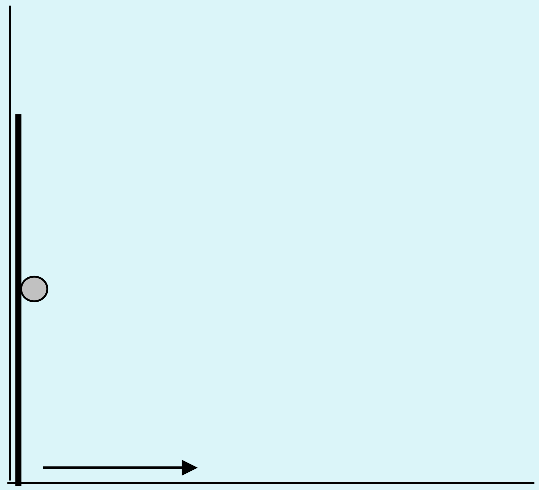
Puzzle: Imagine a cat sitting half way up a ladder that is placed almost flush with a wall.

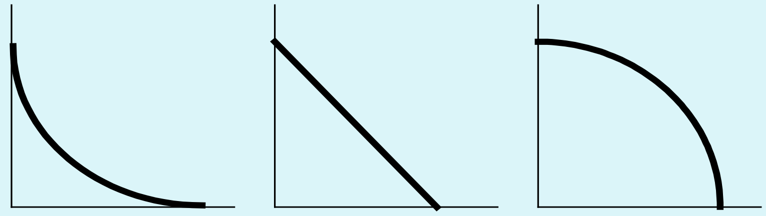


a) If the ladder falls what will the trajectory of the cat be? A, B or C?



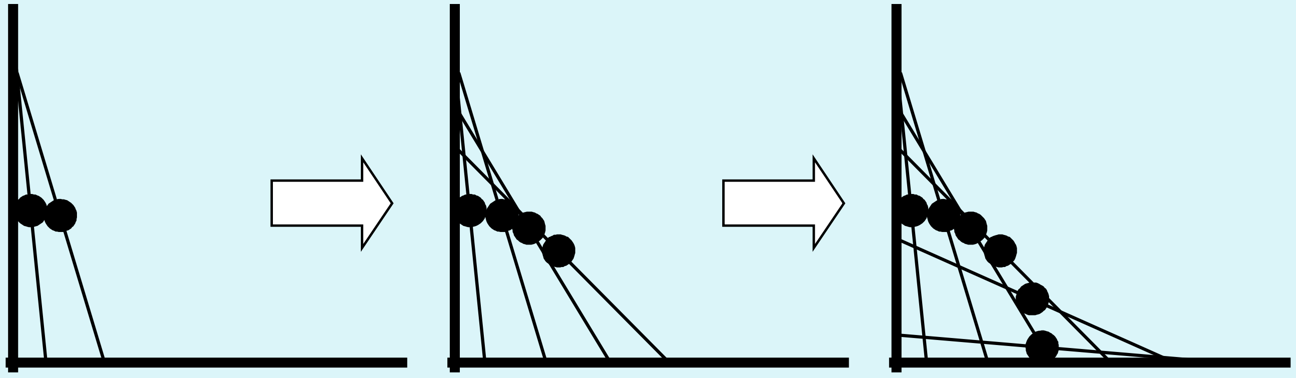
b) Now the *base* is pulled away with the top of the ladder retaining contact with the wall. What will the trajectory of the cat be? A, B or C?

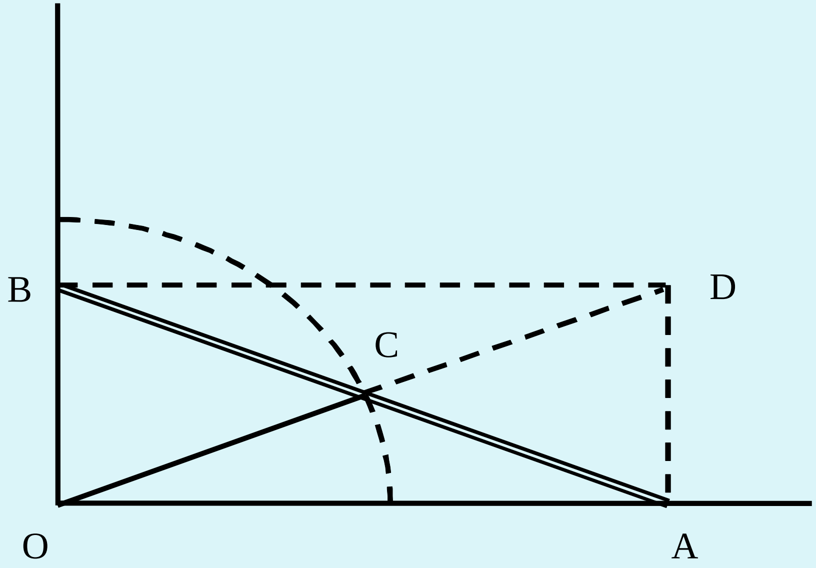




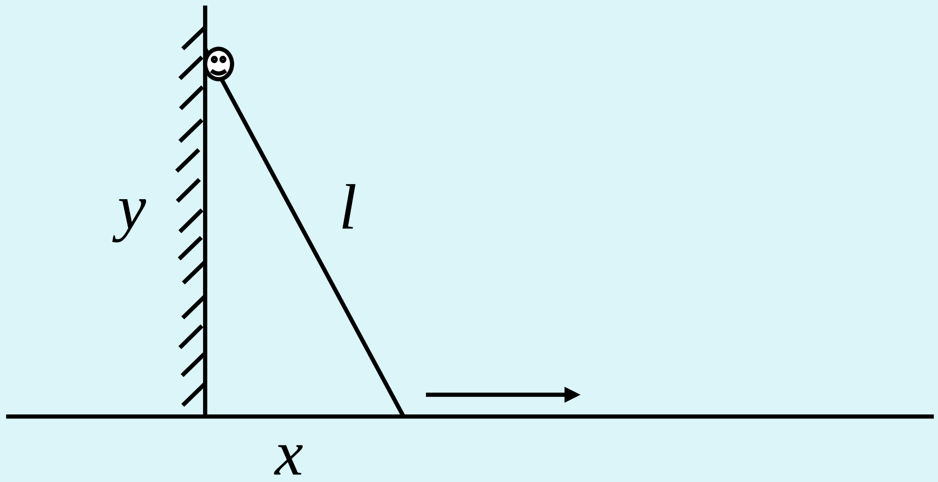
A B C

Still C) !!!





Sophism: Imagine a cat sitting on the top of a ladder leaning against a wall. The bottom of the ladder is pulled away from the wall horizontally at a uniform rate. The cat speeds up, until it's falling **infinitely** fast.



‘Proof’

$y(t)=\sqrt{l^2-x^2(t)}$ where $x(t)$ and $y(t)$ are the horizontal and vertical distances from the ends of the ladder to the corner at time t. Since the ladder is pulled uniformly x' is a constant.

$$y'=-\frac{xx'}{\sqrt{l^2-x^2}} \qquad \lim_{x\to l} y' = \lim_{x\to l}\left(-\frac{xx'}{\sqrt{l^2-x^2}}\right)=-\infty$$

A Wrong Assumption

The ‘proof’ assumes that the ladder maintains contact with the wall while being pulled. This model is simply not true. If all forces involved are considered it can be shown that at one stage the top of the ladder will loose contact and be pulled away from the wall. From that moment the relationship $y = \sqrt{l^2 - x^2}$ is no longer true, since we don’t have a right-angled triangle.

Sophism 1: 0 = 1

Let us find the indefinite integral $\int \frac{1}{x} dx$ by integrating by parts $\int u dv = uv - \int v du$:

$$\int \frac{1}{x} dx = \begin{bmatrix} u = \frac{1}{x} & du = -\frac{1}{x^2} \\ dv = dx & v = x \end{bmatrix} = \left(\frac{1}{x}\right) x - \int x \left(-\frac{1}{x^2}\right) dx = 1 + \int \frac{1}{x} dx$$

Cancelling the integral on both sides we receive 0 = 1.

Sophism 2: 1 = *C*, for any real number *C*

Let us find the indefinite integral $\int \frac{1}{x}dx$ by integrating by parts $\int udv = uv - \int vdu$:

$$\int \frac{1}{x}dx = \begin{bmatrix} u = \frac{1}{x} & du = -\frac{1}{x^2} \\ dv = dx & v = x \end{bmatrix} = \left(\frac{1}{x}\right)x - \int x\left(-\frac{1}{x^2}\right)dx = 1 + \int \frac{1}{x}dx$$

From here $\ln|x| + C_1 = 1 + \ln|x| + C_2$ or 1 = C_1 - C_2 or 1 = C since the difference of two arbitrary constants is an arbitrary constant.

Sophism 3: 1 = *C*, for any real number *C*

Let us apply the u-substitution method to find the indefinite integral $\int \sin x \cos x \, dx$ using two different substitutions:

(a) The substitution $u = \sin x$ with $du = \cos x \, dx$, gives

$$\int \sin x \cos x \, dx = \int u \, du = \frac{u^2}{2} + C_1 = \frac{\sin^2 x}{2} + C_1.$$

(b) The substitution $u = \cos x$ with $du = -\sin x \, dx$, gives

$$\int \sin x \cos x \, dx = -\int u \, du = -\frac{u^2}{2} + C_2 = -\frac{\cos^2 x}{2} + C_2,$$

where C_1 and C_2 are arbitrary constants. Equating the right-hand sides in (a) and (b) we obtain

$$\frac{\sin^2 x}{2} + C_1 = -\frac{\cos^2 x}{2} + C_2$$

Multiplying by 2 and simplifying we obtain $\sin^2 x + \cos^2 x = 2C_2 - 2C_1$ or $\sin^2 x + \cos^2 x = C$, since the difference of two arbitrary constants is an arbitrary constant. On the other hand we know the trigonometric identity $\sin^2 x + \cos^2 x = 1$. Therefore $1 = C$.

Sophism 4: 1 = 0

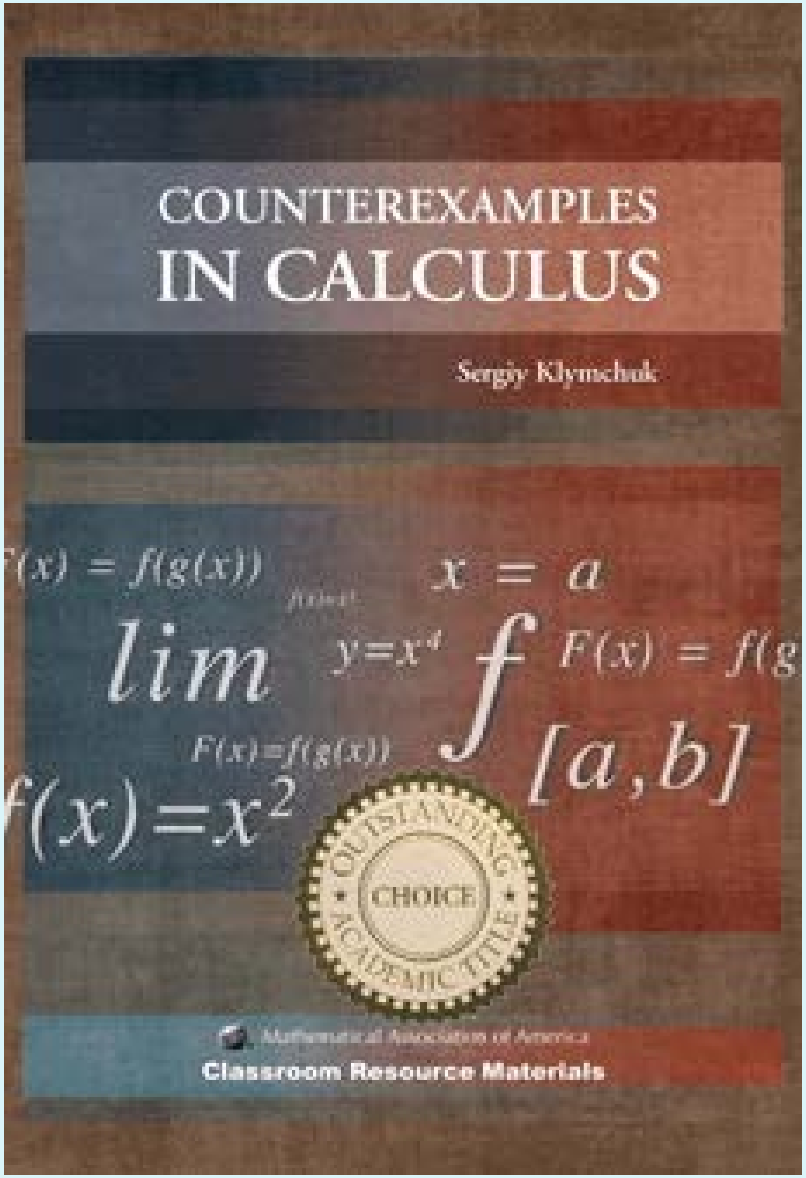
a) On the one hand, for n sequences $\frac{1}{n}$ we have

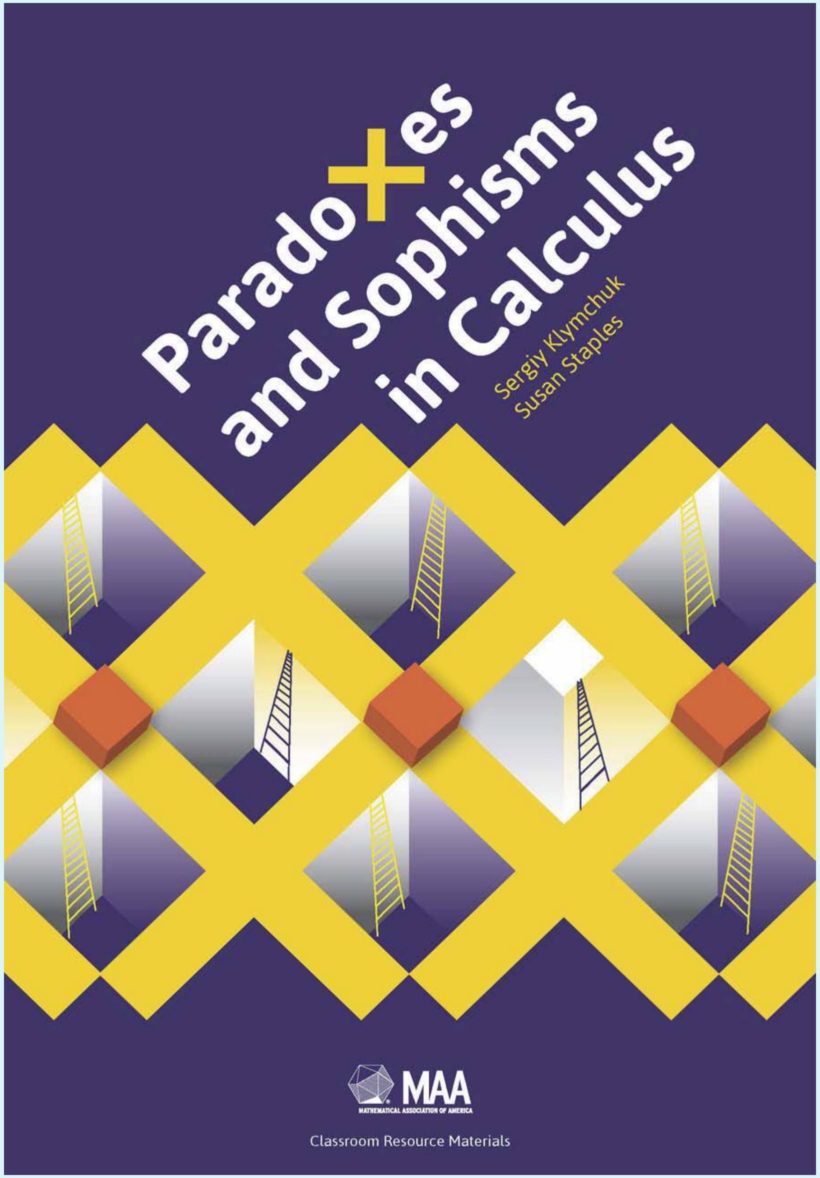
$$\lim_{n\to\infty}\left(\frac{1}{n}+\frac{1}{n}+\ldots+\frac{1}{n}\right)=\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{1}{n}+\ldots+\lim_{n\to\infty}\frac{1}{n}=0+0+\ldots+0=0$$

b) On the other hand, the sum

$$\left\{\frac{1}{n}+\frac{1}{n}+\cdots\frac{1}{n}\right\}_{(n\text{ terms})}$$ is equal to $n\times\frac{1}{n}=1$

c) So, $1 = 0$.

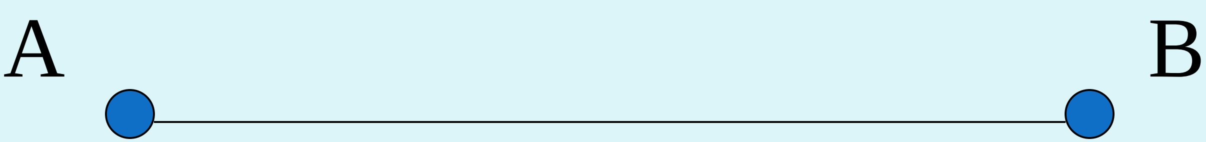

COUNTEREXAMPLES
IN CALCULUS
Sergiy Klymchuk
lim
$x = a$
$y=x^4$
$[a,b]$
$f(x)=x^2$
CHOICE
OUTSTANDING ACADEMIC TITLE
Classroom Resource Materials


Paradoxes
and Sophisms
in Calculus
Sergiy Klymchuk
Susan Staples
MAA
Classroom Resource Materials

Puzzle: Average Speed

You drive a car at a constant speed of 40 km/h from **A** to **B**, and on arrival at **B** you return immediately to **A**, but at a higher speed of 60 km/h.

What was your average speed for the whole trip?



Puzzle: Average Speed

Average speed = Total distance / Total time

Try the distance between A and B as 120 km.

48 km/h

Puzzle: Average Speed

Suppose that you go from **A** to **B** at a constant speed of 40 km/h. What should your constant speed be for the return trip from **B** to **A** if you want to obtain the average speed of 80 km/h for the whole trip?

Puzzle: Average Speed

It is impossible! Even if you drive back from B to A at the speed of light your average speed for the whole trip still would be less than 80 km/h.

Consider the distance between A and B as 40 km.

Average speed = Total distance / Total time

Average speed of 80 km/h = 80 km / (1 h + … h)

Provocations

- Find the derivatives of the functions

  a) $y = \ln(2\sin(3x) - 4)$ b) $y = \ln\ln\sin x$

- Show that the equation $\frac{x^2 + \sqrt{x} + 1}{x - 1} = 0$ has a solution on the interval [0, 2].

  Prove the identity $\sin x = \sqrt{(1 - \cos^2 x)}$.

Pilot Study

- Second year engineering mathematics course
- 6 weeks: 2 puzzles a week during a break between 2 consecutive lectures
- Voluntary participation
- 62 responses out of 65 (response rate 95%)

Questionnaire

Question 1. Do you feel confident solving puzzles?

a) Yes b) No Please give the reasons:

Question 2. Can solving puzzles enhance your problem solving skills?

a) Yes In which way?

b) No Why not?

Question 3. From you point of view, what are the main differences between puzzles and routine problems/questions?

Results

Question 1. Do you feel confident solving puzzles? Please give the reasons.

Yes – 69%

- I have fair idea at times
- Because I am smart
- Because I can
- I am good at problem solving
- I love solving puzzles

No – 31%

- I overthink the problems
- I need more examples to understand the way how to do the question
- Too hard; can get confusing
- I feel that there will always be a catch
- I tend to overthink and overcomplicate everything
- I am constrained by knowledge taught by school system

Results

Question 2. Can solving puzzles enhance your problem solving skills?

Yes – 98% In which way?

- Helps your brain to think more logically and becomes challenging
- Make you look at problems from different angles
- Broadens mind for alternative solutions
- Think in a different perspective, outside the box
- Showing that thinking differently can have amazing results
- Make me think creatively, not always relying on conventional/trained ways of problem solving
- Puzzles place an emphasis on HOW you tackle the problem
- Ability to think with multiple perspectives
- It allows me to come to a solution faster
- Need more puzzles in all maths papers ☺

No – 2% Why not?

- Too different

Results

Question 3. From you point of view, what are the main differences between puzzles and routine problems/questions?

- Puzzles are more fun to solve; more enjoyable and interesting
- Puzzles are more challenging because of the flexibility in approaching
- Puzzles require creative thinking and more careful reading
- Puzzles add a bit more variety; are more tricky, freshen up your mind
- Puzzles require more insight, creativity; more thinking and novel solutions
- Puzzles aren’t always straight forward, some people just can’t get them no matter what
- A puzzle requires us to throw away those old/stubborn stuff in my brain in order to solve it
- Puzzles relate to more realistic things
- Puzzles are exciting and help to keep me alert
- Puzzles set a more fun environment compared to routine problems
- Puzzles test your problem solving skills and routine problems are testing if you can follow problems

Conclusions

- About 2/3 of the students reported that they were confident in solving puzzles and 1/3 that they were not
- Almost all students (61 out of 62) do believe that solving puzzles enhances their problem solving skills
- It gives us confidence to continue with the LTDF project
- Evaluate the relationship between the ability in solving puzzles and course performance
- Investigate the effect of using puzzles on student engagement, in particular attendance
- Analyse the impact of using puzzles on student decision to continue their study

Thank you for your attention