A New Type of Question for Teaching and Assessing Critical Thinking in Mathematics

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Report on Sabbatical, Sem-2 2020

- Published 10 PBRF-rated research outputs including 9 quality assured;
- Produced the final report as Principal Investigator of the large national project supported by a $198,205 grant from the New Zealand Council for Educational Research;
- Reviewed articles for 2 international research journals: *International Journal of Mathematics Education in Science and Technology* and *International Journal on Teaching Mathematics and its Applications*;
- Finalised a joint paper with my collaborator Professor Chris Sangwin, University of Edinburgh to be submitted to an international Q1 journal;

Report on Sabbatical, Sem-2 2020

- Led the STEM-TEC Centre - budgeting, securing seminar speakers, updating the website, online communication with members, producing annual report, running meetings on campus;
- Supervised PhD student Kerri Spooner as primary supervisor – regular online and face-to-face meetings;
- Developed MATLAB tasks for the recently reviewed paper ENGE601 Engineering Mathematics-2;
- Participated in Mathematical Sciences review/refresh – preparing two new paper descriptors, attending all meetings.

Publications

- 3 refereed international journal articles including 2 in a Q1 journal;
- 1 journal article published in hard copy that was published earlier online;
- 4 refereed international conference papers;
- 1 refereed national research project report;
- 1 plenary address at the Auckland Mathematical Association conference.

Theme-1

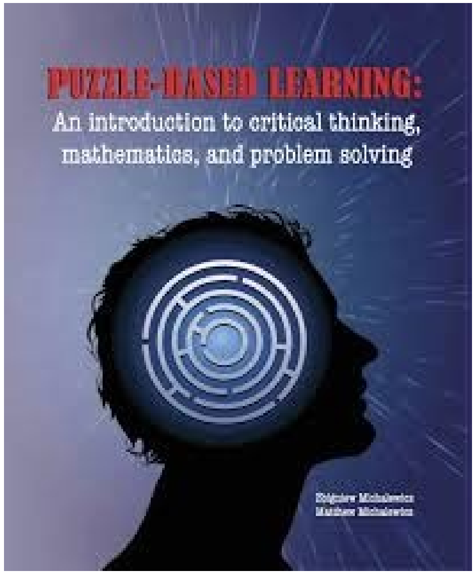
Evaluating the Impact of Puzzle-Based Learning on Creativity, Engagement and Intuition of STEM Tertiary Students

- A national 2-year project supported by a large grant from the New Zealand Council for Educational Research
- Team: 10 academics from 4 tertiary institutions (AUT, UoA, MIT, Whitireia Polytechnic)

Introduction

In recent years, some universities in Australia, Europe and the USA, have introduced formal academic courses (in some universities compulsory) for their first-year STEM students based on a Puzzle-Based Learning (PzBL) pedagogical strategy promoted in the book

Michalewicz, Z., & Michalewicz, M. (2008). *Puzzle-based learning: An introduction to critical thinking, mathematics, and problem solving.*



Introduction

- The feedback reported in several pilot studies was very positive
- It indicated that PzBL could potentially increase students' engagement and enhance their creativity, general problem-solving skills and lateral thinking "outside the box".
- However, the feedback was primarily based on students' and lecturers' attitudes and perceptions.
- In this study, we attempted to measure the impact of the PzBL on students' creativity, engagement and intuition

Puzzles

- A puzzle is a non-routine and not standard problem presented in an entertaining way.
- It looks deceptively simple but often has an unexpected answer and a surprised counterintuitive solution.
- Solving a puzzle requires creativity: thinking ‘outside the box’.
- Many high-tech companies use puzzles at their job interviews to evaluate candidate’s innovative and creative problem solving skills and select the best of the best.

Puzzles

Puzzles illustrate many powerful general problem solving principles, techniques and strategies such as:

- extreme values principle
- start at the end
- guess and check (trial and error)
- sketching and/or modifying a diagram
- looking from another angle (literally and metaphorically)
- splitting into smaller parts
- solving an analogous case first
- looking for a pattern
- invariance principle (something stays the same)
- using easier numbers

Example 1.

A rectangular chocolate bar consists of 10×6 small rectangles and you wish to break it into its 60 constituent parts. At each step, you can only pick up one piece and break it along any of its vertical or horizontal lines. How should you break the chocolate bar using the *minimum* number of steps (breaks)? What is the *minimum* number?

Answer: 59

After the first break you have 2 pieces, after the second break you have 3 pieces, after the third break four pieces, and so on. The number of breaks is always one less than the number of pieces. To break into 60 pieces, you need 59 breaks regardless of the way you break the chocolate. It is an illustration of the invariance principle.

Example 2.

Fifty five players start a tennis tournament. How matches will be played if a player who loses a match leaves the tournament.

Answer: 54

To select the winner at the end, 54 players need to be eliminated so it needs 54 matches. It is an illustration of the ‘start at the end’ problem-solving strategy.

The Study

- PzBL intervention: one semester
- About 700 students - Year 1 and Year 2 - studying 12 courses in astronomy, computing, engineering, and maths
- Regular use of puzzles in lectures as a co-curricular activity over a 12-week semester.
- Typically, 3–4 puzzles every week: 5-7 minutes a week
- Not part of assessment
- Instruments: student pre- and post-test questionnaires, student focus group interviews, lecturer questionnaires, and class observations
- Statistical analyses: SPSS 25 program incl. Repeated Measures ANOVAs, Friedman test, regression analysis.

Actual Outcomes

- Students' behavioural engagement was significantly greater during the intervention
- Behavioural and cognitive engagement are often used as indicators of improved learning
- The group with C grades in prerequisite courses appeared to be the most engaged with non-routine problem-solving.
- It has the potential to increase the student retention rate.
- Even though students saw the importance of inhibiting intuitive thinking, it did not change significantly over time.
- There were no significant changes in students' creativity

Actual Outcomes

- The intervention may have been more effective for males than females - could be investigated in future research.
- Students agreed that solving non-routine problems was useful for their learning and was able to enhance their creative and innovative thinking abilities.
- Perceptions of the utility value of non-routine problem-solving improved at the end of the semester for all students.
- The students strongly agreed that solving non-routine problems in their courses would be beneficial to their future learning, as well as their careers and other areas of life.

Theme-2

University Students' Preferences for Application Problems and Pure Maths Questions

A survey of year-1 and year-2 engineering students

- Which questions do you prefer to solve: application problems or pure maths questions? Please circle one:

  Application problem   Pure maths question

- Why? Please circle all that apply:

- More relevant
- More interesting
- More useful
- Easier
- Other (please specify)

Examples

An application problem:

The rate of decay of the mass of a radioactive substance is proportional to the mass of the substance present. Initially the mass of the substance was 165 mg. At the end of 1.5 years, 112 mg remain. How much will there be after 3.5 years?

An equivalent pure maths question:

Find the solution of the initial value problem $\frac{dy}{dx} = ky$, $y(0) = 165$ at the point $x = 3.5$ provided $y(1.5) = 112$.

An application problem:

An electric circuit consists of an inductor (with inductance $L = 1$ henry), a capacitor (with capacitance $C = 0.0625$ farads) and a power supply $E(t) = 8$ volts at time t seconds, $t = 0$. No current is flowing in the circuit at time $t = 0$. Use Laplace transforms to find $i(t)$—the current in this circuit at time t. Use the Kirchoff's second law: $L\frac{di}{dt} + \frac{1}{C}\int_0^t i(\tau)\,d\tau = E(t)$.

An equivalent pure maths question:

Use Laplace transforms to solve the integro-differential equation $\frac{dy}{dx} + 16\int_0^x y(s)\,ds - 8 = 0$ with the initial condition $y(0) = 0$.

Surprising Results

- More than half of the engineering students preferred pure maths problems to application problems: 43 vs. 37
- Students' preference in pure maths problems most likely was dominated by the common belief that pure maths version of an application problem is easier to solve.
- It was reported by 25 students compared to only 5 students who indicated that applications problems are easier for them than pure maths questions.
- This is in spite of the fact that students found application problems more relevant to them (17 vs. 12), interesting (19 vs. 2) and useful (12 vs. 11).

Surprising Results

Table 1. *Students' preferences for solving problems in engineering mathematics courses*

Reasons	Prefer application problems			Prefer pure maths questions		
	Group 1	Group 2	Both	Group 1	Group 2	Both
	25	12	37	26	17	43
More relevant	10	7	17	5	7	12
More interesting	12	7	19	2		2
More useful	6	6	12	6	5	11
Easier	2	3	5	11	14	25
Other				1	1	2

Theme-3

A New Type of Question for Teaching and Assessing Critical Thinking in Mathematics

Rationale

- Making mistakes is a natural part of every human activity
- The reality is that mistakes, errors, incorrect statements and misinformation are everywhere
- Most of them are unintentional but some are deliberate and made to mislead, misguide, and misinform like for example fake news
- Some of them have a significant impact on individuals and organizations and contribute to so-called 'information wars' between conflicting countries

Rationale

- Regardless of the nature of a mistake, an ability to spot a mistake and avoid becoming the victim of it is a valuable skill and a key aspect of critical thinking:

- identifying contradictory information;
- eliminating impossible cases;
- recognising mistakes, errors, inconsistences;
- using sceptical and unbiased analysis;
- paying attention to detail;
- making rational judgement and decisions based on factual evidence

Mistakes in Maths

- Mathematics is not immune from mistakes

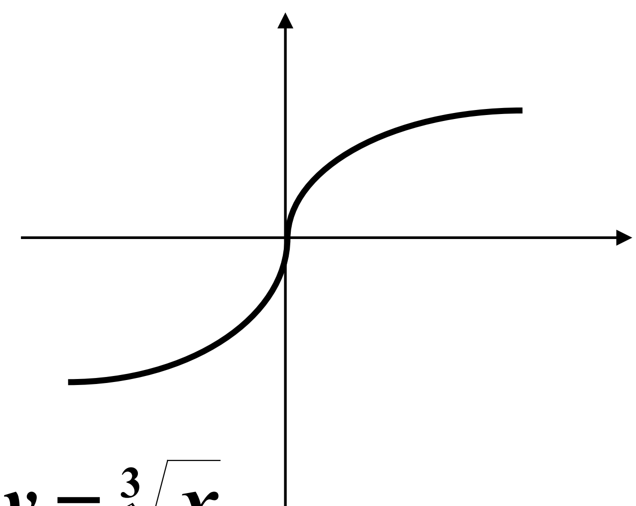
- Mistakes are in:

- maths textbooks
- dictionaries of maths
- research journals on maths education
- national school maths exams

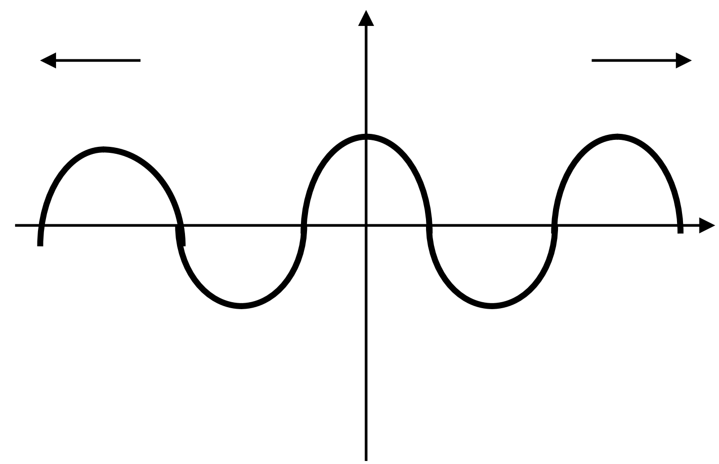
Example 2 (Textbook)

Bolton, W. (1997). *Essential Mathematics for Engineering.*

"With a continuous function, i.e. a function which has values of y which smoothly and continuously change as x changes for all values of x, that we have derivatives for all values of x." (p. 332)



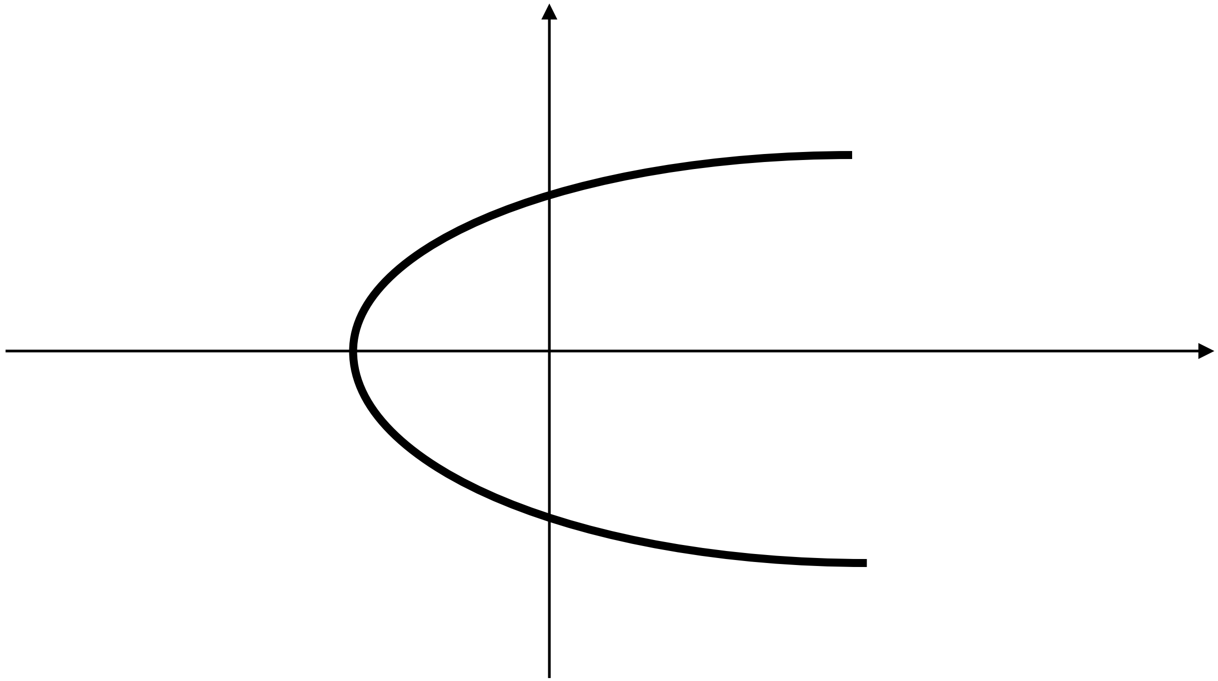
$y = \sqrt[3]{x}$



Example 3 (Dictionary of Maths)

Borowski, E., Borwein, J. (1989). *Collins Reference Dictionary of Mathematics*.

"The function $y = \sqrt{(x+2)}$ whose graph is shown below…" (p.132)



Example 4 (Top Journal in Maths Ed)

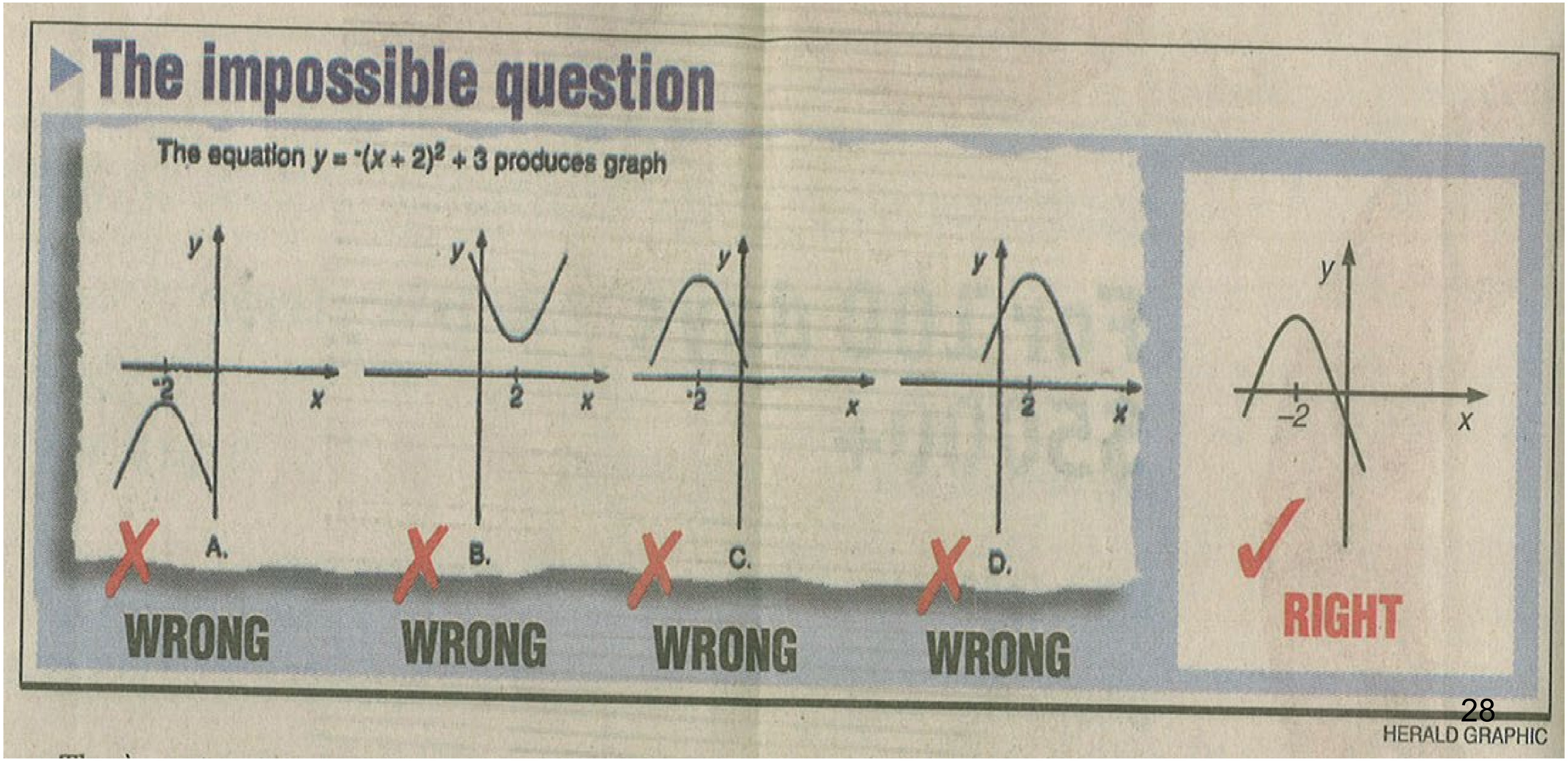
International Journal *Educational Studies in Mathematics* (1998). v.35(2).

"For both equations $x^2 + 2x + 1 = 0$ *and* $\sin^2 x = 1$ the equal symbol means equality for only 2 values of x." (p.155).

The first equation is valid for one ***value*** of x and the second for infinitely many ***values*** of x.

Example 5 (NZ Exam in Maths - 2000)

$$y = -(x+2)^2 + 3$$



Recent NZ Exams

- "The National Party is calling for a review of how senior school exams are set after yet another mistake emerged in a maths paper. The latest mistake in an algebra question in the Level 2 exam for the National Certificate of Educational Achievement (NCEA) in 2019 follows five mistakes in maths and statistics exams in 2016"

  The New Zealand Herald, 24 November 2019

- The students were "baffled", "shocked" and "confused" by those mistakes and in some cases "disadvantaged"

Example 6 (Newspaper)

А КАК У НИХ

Какую скидку можно получить на зарубежных курортах

Египет. Торговаться можно практически в любой торговой точке. Можно уходить и возвращаться. Главное - не жалеть времени, потому что торг тут - часть местного колорита. Цену можно снизить на 100%-150%.

Discount at holiday resorts

<u>Egypt.</u> You can bargain at any shop. It's a part of local culture and customs. You can reduce the price by 100%-150%.

Example 7 (Politics)

"317 MILLION PEOPLE IN AMERICA AND YOU SPEND

360 MILLION ON JUST INTRODUCING OBAMACARE?

JUST GIVE EACH CITIZEN A MILLION BUCKS!"

Parker, M. (2019). *Humble Pi: A Comedy of Maths Errors.*

Example 8 (Transport)

“The ship is facing the wrong way and will need a 360° turn before sliding through the relatively tiny harbour entrance”

(MisMaths website, a collection of mistakes, misconceptions and misrepresentations involving mathematics in media)

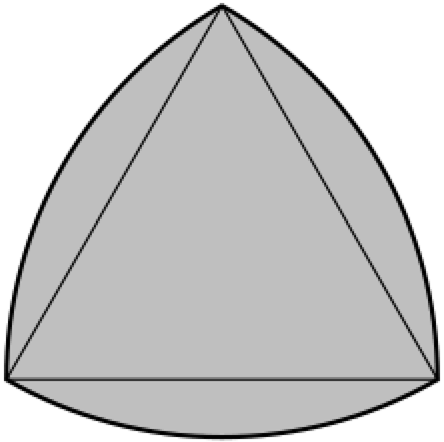
Example 10 (Challenger Tragedy)

"The Challenger exploded because of a leak from one of the solid rocket boosters… The performance of the rubber O-rings was definitely the primary cause of the accident and remains the headline finding that most people remember… But Feynman also uncovered a second problem with the seals between the booster sections, a subtle mathematical effect... Checking if a cross-section of a cylinder is still circular is not that easy. For the boosters, the procedure for doing this was to measure the diameter in three different places and make sure that all three were equal." pp.75-77

Parker, M. (2019). *Humble Pi: A Comedy of Maths Errors.*

Example 10 (Challenger Tragedy)

- The NASA engineers used the wrong criterion of a constant diameter to identify a circular shape
- The converse of the statement “if we have a circle then the diameter is constant” is false, for example the famous Reuleaux triangle, a curve of constant width



- That mathematical error contributed to the disaster as reported in Finding #5 of the Investigation Report

The Proposal

To prepare students for real life better it is proposed to include so-called provocative mathematics questions in teaching and assessment

Definition

- A provocative mathematics question is a question that is *deliberately* designed to mislead the solver
- It looks like a typical routine ‘innocent’ task but in fact it has a catch
- It normally requires to do an impossible task
- Often it is based on a restricted domain or indirectly prompts the use of a rule, formula, or theorem that is inapplicable due to their conditions/constraints

Why Provocations?

- To prepare students for real life by enhancing their critical thinking including the abilities to analyse questions and recognize mistakes

- The intention is that students transfer/apply these skills outside mathematics so they become better-informed citizens and critical thinkers

- Critical thinking is one of the top skills required by employers

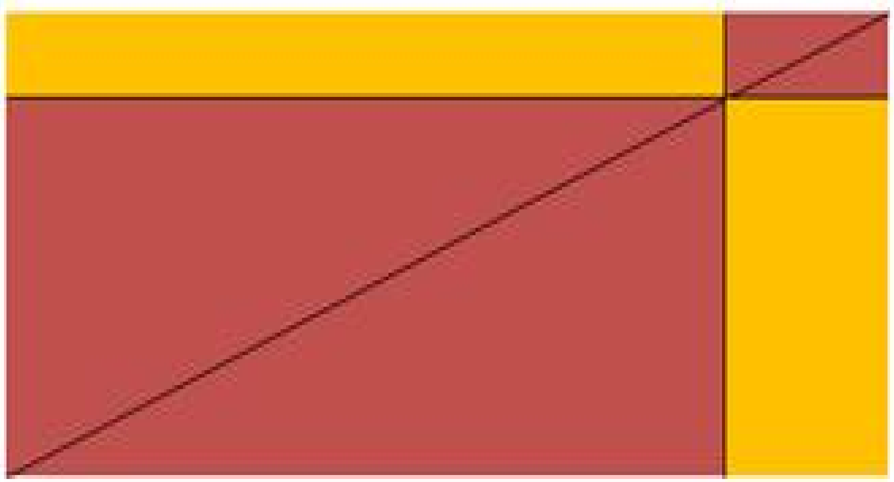
Future of Jobs, World Economic Forum (2016): Top 10 Skills in 2020

1. Complex Problem Solving
2. Critical Thinking
3. Creativity
4. People Management
5. Coordinating with Others
6. Emotional Intelligence
7. Judgment and Decision Making
8. Service Orientation
9. Negotiation
10. Cognitive Flexibility

Examples of Provocative Questions

1. Show that the area of the triangle with the sides 20 cm, 10 cm and 8 cm can be larger than 50 sq.cm

2. Prove that the orange rectangle on the right has a larger area than the orange rectangle on the left



Examples of Provocative Questions

3. A man has 30 pigs and orders that they are to be killed in 3 days, an odd number each day. What odd number of pigs must be killed each day?

   (Alcuin of York, born 732 AD, “Problems to Sharpen the Young”)

4. Sketch a graph of a function that is differentiable on (a, b) and discontinuous at least at one point on (a, b)

5. Find the derivative of the function $y = \ln(\ln(\sin x))$

Possible but Provocative

6. Linda Problem. Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations. Which alternative is more probable?

a) Linda is a bank teller.

b) Linda is a bank teller and is active in the feminist movement.

(Daniel Kahneman, “Thinking, Fast and Slow”)

Study-1

Five groups of school mathematics teachers (total 127) from:

- New Zealand (14 teachers)
- Hong Kong (26 teachers)
- Germany (10 teachers)
- Ukraine (26 students - future teachers)
- Australia (51 teachers)

Mini-test

Some of the following 7 questions are just routine questions and some have a catch.

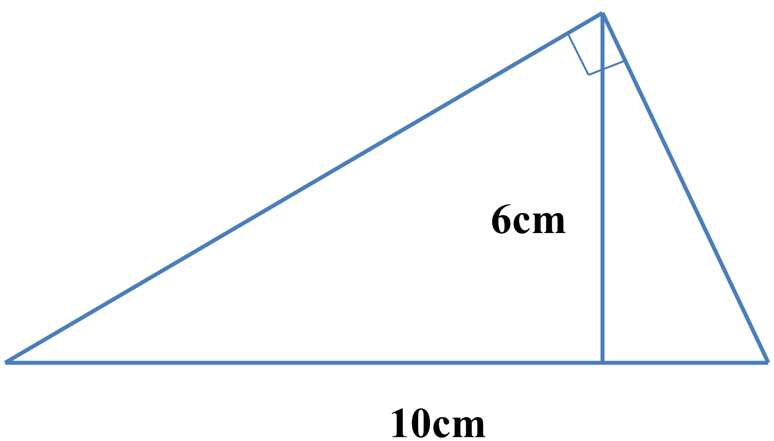
You decide which is which.

Please solve all 7 questions.

You have 15 minutes for this mini-test.

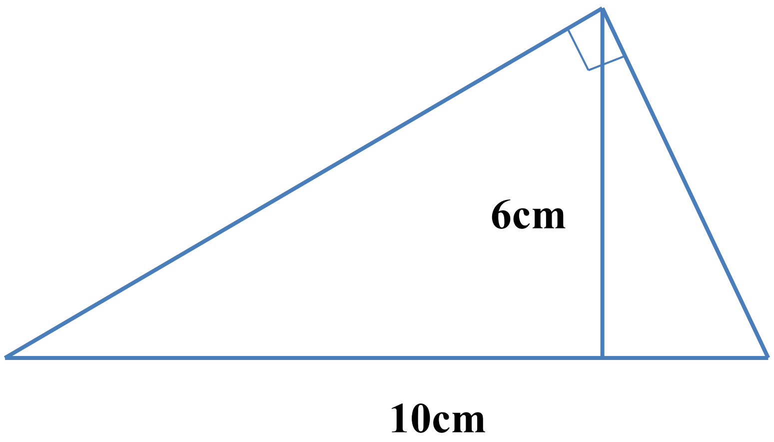
Examples of Questions

Question 1. Find the area of the right-angled triangle if its hypotenuse is 10cm and the height dropped on the hypotenuse is 6cm.



Answer

<u>Question 1.</u> Find the area of the right-angled triangle if its hypotenuse is 10cm and the height dropped on the hypotenuse is 6cm.



0 (the triangle doesn't exist)

Examples of Questions

Question 4.

Prove the identity $\sin x = \sqrt{(1 - \cos^2 x)}$.

Answer

Question 4.

Prove the identity $\sin x = \sqrt{(1 - \cos^2 x)}$.

This is not an identity but an equation with infinitely many solutions

Examples of Questions

Question 6.

Find the derivative of the function

$$y = \ln(2\sin(3x) - 4).$$

Answer

<u>Question 6.</u>

Find the derivative of the function

$$y = \ln(2\sin(3x) - 4).$$

***y*' doesn't exist as *y* doesn't exist**

Results of the Mini-test

Correct answers	Q1.	Q2.	Q3.	Q4.	Q5.	Q6.	Q7.
Group 1	0%	7%	21%	7%	0%	8%	0%
Group 2	23%	12%	27%	19%	12%	15%	12%
Group 3	0%	60%	30%	N/A	20%	0%	0%
Group 4	0%	19%	31%	15%	12%	0%	8%
Group 5	2%	54%	33%	4%	28%	9%	28%

Questionnaire

Question 1. What are your feelings after you have learnt about the correct solutions to the test questions?

Question 2. What are the reasons for not solving all test questions correctly?

Question 3. Would you make any changes in your teaching practice after doing the mini-test? If so – which changes? If not – why?

Question 3 (changes)

New Zealand (14) and German (10) groups:
All participants reported that they would make changes in their teaching practice.

"Introduce tricks like this to class to make them think; encourage and reward checking of answers; more emphasis on the validity of solutions; teach them to examine the question thoroughly; give students more questions that will force them to think about the conditions surrounding the questions; encourage students to think through questions carefully; give students questions to challenge their knowledge; stop answering impulsively, think before respond; expose students to such questions to get them to think more deeply about the conditions."

Question 3 (changes)

Hong Kong (26), Ukrainian (26) and Australian (51) groups:

About half of the participants reported that they would make changes in their teaching practice.

The other half reported that the questions from the test were not part of assessment and therefore they would not change their teaching practice: “exams always have questions which make sense, so why teach them beyond the process?”, “I am worried that in an assessment they will become absorbed by looking for the trick and waste precious time, as the assessments they do, do not have trick questions”, “in my teaching practice my students follow the script”, “we are trying to get them to be successful in their exams after all.”

Australian Group only

- “I think this is an ‘immoral’ test. I don’t believe you should ask someone to prove something is true when it isn’t”
- “Make sure questions work and don’t be nasty”
- “Students should be able to trust the questions”
- “When students are doing a test they shouldn’t be looking for a trick all the time. It is a matter of trust”

Typical assessment

- Mostly routine questions where all conditions of formulas/rules are satisfied
- Students might develop a habit of applying formulas and rules without checking the conditions/constraints
- In real life not everything behaves so nicely and ignoring conditions/constraints might lead to significant and costly errors

NZ Exam: Question 1

Question 1. Show that the equation $x^2 - \sqrt{x} - 1 = 0$ has a solution between $x = 1$ and $x = 2$.

Model Solution. "If $f(x) = x^2 - \sqrt{x} - 1$ then $f(1) = -1 < 0$ and $f(2) = 1.58 > 0$. So graph of f crosses x-axis between 1 and 2."

Why is the condition of continuity ignored?

What message does this send to the students?

Study-2

Two groups of school mathematics teachers (total 82) who attended my talks on provocative questions in 2019:

- AMA Maths and Calculus Day
  November-2019 (43 teachers)

- NZMS Colloquium Maths Teachers Day
  December-2019 (39 teachers)

Questionnaire

Question 1. Would you agree to use provocative questions in teaching/learning of maths? Why?

Yes (96%) No (4%)

Main categories of the comments to Yes:

1. Enhancing critical thinking
2. Deeper understanding of maths concepts
3. Promoting creativity, engagement and interest
4. Benefits outside classroom
5. Gradual introduction of provocative questions and possible support especially for weaker students

Questionnaire

Question 2. Would you agree to use provocative questions in assessment? Why?

Yes (63%) No (37%)

Main categories of the comments to Yes:

1. Students' benefits and the need to assess what is taught
2. The need for students' prior experience and practice with provocative questions
3. Connections to the real world

Questionnaire

Main concerns of those who anwered No:

1. Potential additional stress for students (56%)

"Generate confusion and stress; would make our students too anxious although would benefit the brighter students; would increase maths anxiety in most students; students are already under lots of stress; students are already stressed about sitting assessment; students are already stressed about their grades; many students are already nervous; my students would revolt!"

Questionnaire

Main concerns of those who anwered No:

2. Students’ preparedness for provocative questions (44%)

“Too soon, need to prepare the students, maybe one day; maybe with top students, would need more time to develop this; not possible at this stage under NCEA; not yet, maybe later once I’ve seen how things go out in teaching; students not ready for that; not yet, students may suspect many questions are ‘provocative’ and give up trying to solve them; not yet, needs introducing slowly”

Discussion

- Sometimes erroneous statements are being used in the assessment.
- However, normally they are clearly labelled as incorrect statements.

- For example:

- critique the given misleading graph;
- disprove the wrong statement by a counterexample;
- identify a mistake in the sophism;
- verify whether the given statement is correct or incorrect

Discussion

- But such questions contain prompts and hints that something is wrong and this alerts students to think critically and act differently compared to solving a standard procedural question.
- However, in real life there are often no prompts and hints about coming errors.
- Therefore, the provocative questions go one step further.

Discussion

- Lots of practice in classroom as with other three types of questions - procedural, conceptual, applications – so students get broad experience
- The inclusion of provocative questions should be gradual:

  - first use them as an additional, extracurricular activity in a classroom;

  - then include them in the mathematics curriculum and subsequently into formative assessment;

  - finally include them into summative assessment

Discussion

- The following warning/hint should be added to the instruction page of the exam: “Some of the questions in this exam are typical routine questions and some are deliberately designed to mislead you. You should decide which is which and act accordingly”.

- A possible side effect - apart from enhancing critical thinking such questions might also increase students’ motivation and engagement in a classroom and make mathematics more attractive and meaningful

Discussion

- Other subjects, e.g. physics

  Example. Give an example of the most famous quadratic equation in physics

- Many people would answer $E = mc^2$.

- The answer is incorrect: it is not a quadratic equation, as c is a constant, so it is a linear equation.

Discussion

- Practice in solving and designing provocative questions should be an integral part of training of prospective mathematics teachers, and is included into professional development of in-service school mathematics teachers
- The investment in training is very small – attending just 1-2 workshops – however the benefits for the students and society is enormous

Future of Jobs, World Economic Forum (2016): Top 10 Skills in 2020

1. Complex Problem Solving
2. Critical Thinking
3. Creativity
4. People Management
5. Coordinating with Others
6. Emotional Intelligence
7. Judgment and Decision Making
8. Service Orientation
9. Negotiation
10. Cognitive Flexibility

Education vs Training/Drilling

"*Education is what remains after one has forgotten everything he has learnt*".

Albert Einstein

Thank you

NCEA Level 2 Algebra Exam-2019
A Perfect Provocative Question!

The shape below is divided into rectangles. All measurements are in cm.

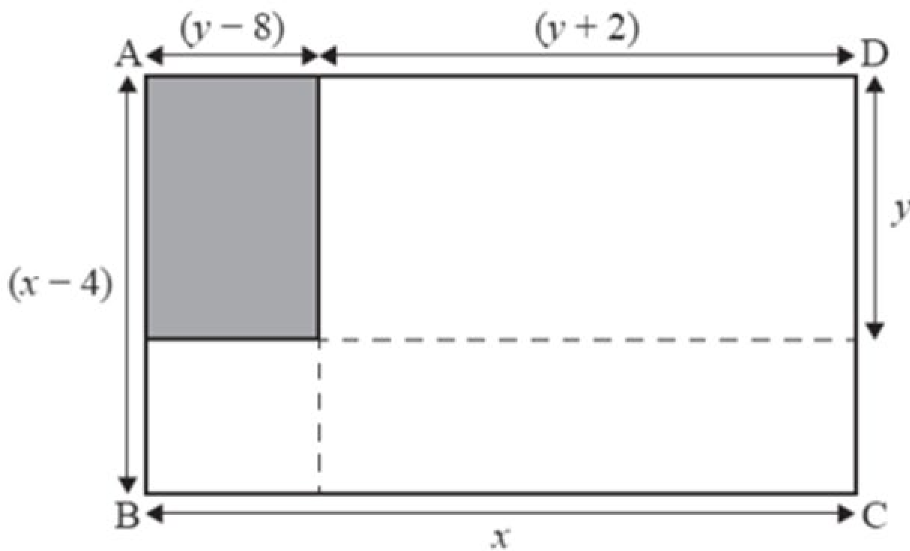


Diagram is NOT to scale

The shaded rectangle has an area of 9 cm^2.

Find the area of the rectangle ABCD.